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On the Essentiality of Credit and Banking at Zero Interest Rates*

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Abstract

We investigate the welfare-increasing role of credit and banking at zero interest rates in a microfounded general equilibrium monetary model. Agents differ in their opportunity costs of holding money due to heterogeneous idiosyncratic time-preference shocks. Without banks, the constrained-efficient allocation is never attainable, since impatient agents always face a positive implicit rate in equilibrium. With banks, patient agents pin down the borrowing rate and in turn enable impatient agents to borrow at no cost when the inflation rate approaches the highest discount factor. Banks can therefore improve welfare at zero rates, provided that both types of agents are included in the financial system and that the borrowing limit is sufficiently lax. The result is robust to several extensions.

Keywords: Banking, Money, Zero Interest Rates

JEL codes: E40, E50

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1 Introduction

A common result in the theoretical monetary literature finds that credit does not increase welfare when nominal interest rates are set to zero, a policy known as the Friedman rule.[1] The logic behind this result is that zero interest rates eliminate the opportunity cost of holding money. Thus, money enables agents to perfectly self-insure against consumption risk and the equilibrium allocation is the same with or without credit. That is, credit is not essential at zero rates. An underlying assumption of this argument is that agents are homogeneous. There is, however, ample evidence that agents are characterized by different preferences and specifically by different degrees of patience, as shown for example by Lawrance (1991), Samwick (1998), and Falk et al. (2018). But if this heterogeneity does exist, the opportunity cost of holding liquid assets is not the same across individuals and impatient agents will make different portfolio decisions compared to patient ones. Microeconomic evidence supports the existence of these portfolio differences. For example, Weidner, Kaplan, and Violante (2014) show that roughly one-third of US households hold almost no liquid assets. Does this heterogeneity lead to a welfare-enhancing role of credit and banking, even when interest rates are zero? Under what conditions? The purpose of this article is to address these questions.

We conduct our analysis using a microfounded general equilibrium model of banking where money has an explicit role. The presence of both money and banks here is important since we want to understand if banks change the allocation compared to an economy where money is the only asset. Specifically, our study builds on the monetary model of Berentsen, Camera, and Waller (2007), where perfectly competitive banks accept nominal deposits and make nominal loans. In their set-up, banks have access to a record-keeping technology that allows them to keep track of financial histories but not of goods-market trades. Therefore, the existence of financial record keeping does not eliminate the need for money as a medium of exchange. In their framework, banks are essential for any positive nominal interest rate because they pay interest on deposits, which in turn decreases the cost of holding idle balances. At zero interest rates, however, the return on money and on deposits is the same and there is no need for financial intermediation.

We enrich the environment in Berentsen, Camera, and Waller (2007) by introducing idiosyncratic shocks to agents' discount factors. These shocks imply heterogeneity in terms of the opportunity costs of holding liquid assets across time, given that in every period some agents are more patient than others. In our environment, equilibrium deflation is bounded by the lowest discount rate, since giving cash a return exceeding the lowest shadow interest rate would generate arbitrage opportunities. As a consequence, interest rates are zero when

1. We review the theoretical monetary literature on this topic in Section 2.

the gross inflation rate approaches the discount factor of the patient agents. When that is the case, money is costless to carry intertemporally for patient agents but it is still costly for impatient ones to do so. Thus, impatient agents are always constrained when they can only use money to trade and the efficient allocation is unattainable at the Friedman rule when cash is the only asset. This result is common to Boel and Camera (2006) and Boel and Waller (2019), whose work displays permanent and temporary heterogeneous degrees of patience, respectively. Neither paper, however, analyzes the role of banking and credit in this environment. In Boel and Camera (2006), agents can save via government bonds and the efficient allocation can be sustained when bonds are illiquid and pay positive yields, but only provided patient agents trade more frequently than impatient ones. In Boel and Waller (2019), stabilization policy in response to aggregate demand shocks temporarily relaxes the liquidity constraint of impatient agents without harming the patient ones and thus improves welfare even at zero rates. Nevertheless, such a policy cannot achieve the efficient allocation in a monetary equilibrium.

Several results emerge from our analysis. First, credit is essential at zero interest rates, meaning that it improves the allocation. In our model, agents can borrow intertemporally via IOUs and intratemporally via banks. In equilibrium, patient agents are the price setters and pin down the interest rate for the IOUs. Due to a non-arbitrage condition between bond and bank rates, patient agents also pin down the bank borrowing rate. This implies that, when the gross inflation rate approaches the highest discount factor, the bank borrowing rate is zero. This in turn means that impatient agents can borrow from a bank at no cost, whereas those same agents would face a positive implicit rate if they only had access to money. This result relies on both patient and impatient agents participating in the same banking system, so that impatient agents can borrow at the same interest rate as patient ones. If this type of financial inclusion occurs, banks improve welfare at zero interest rates provided that the borrowing limit is lax enough, that is, if there are enough depositors.

Second, changes in credit conditions are not necessarily neutral. In equilibrium, the credit limit depends on the amount of deposits. When the borrowing constraint is tight, money and credit are perfect substitutes for impatient agents. In this case, tightening the debt limit is neutral. As the constraint relaxes, impatient agents prefer not to hold money and simply borrow if needed. This change happens because borrowing is costless at zero interest rates, whereas money is still costly to hold for impatient agents. In this case, increasing the borrowing limit improves the allocation and banks increase welfare. Indeed, if there are sufficient deposits, banks enable the economy to achieve the efficient allocation in a monetary equilibrium.

Third, the reason for banks' welfare-increasing role differs at positive interest rates compared to zero rates. In Berentsen, Camera, and Waller (2007), banks are essential because they pay interest on deposits, thus reducing the cost of holding idle money. In our paper, this is still the role of the bank when the interest rate is positive, but such a function vanishes at the Friedman rule. Indeed, when deposit rates are zero, patient agents are indifferent between holding cash or depositing balances at the bank. The increase in welfare is instead due to banks' ability to relax the liquidity constraints of impatient borrowers. That is, the bank's ability to enforce the repayment is essential at the Friedman rule. If such a technology were available to all agents and they were allowed to make credit arrangements among themselves, they would be able to achieve the same allocation that banks enable. But that would be a non-monetary equilibrium.

The results are robust to several extensions. First, we examine the case where banks face operating costs. Boel and Camera (2020) find that, when banking is costly, banks are welfare increasing only for nominal interest rates bounded sufficiently away from zero. Chiu and Meh (2011) find that financial intermediation is welfare improving only at high inflation, that is, when the inflation rate is sufficiently large relative to the fixed cost. We instead show that banks can improve welfare at zero rates even if costly, but only provided that the borrowing limit is sufficiently high. Second, we investigate whether the coexistence of money and credit at zero interest rates holds for any type of heterogeneity or if instead the result hinges specifically on agents having different time preferences. We find that the heterogeneity in time preferences is crucial, the reason being that different discount factors lead to different shadow rates and banks allow impatient agents to borrow at the same interest rate as patient ones. With other types of heterogeneity, all agents can perfectly self-insure against consumption risk only with money at the Friedman rule because the shadow rate is zero for both types even without banks. Third, we examine an equilibrium where banks cannot force repayment of loans, thus implying that borrowers might have an incentive to default. As inflation influences equilibrium payoffs, it in turn affects the endogenous borrowing limit. We find that money and credit do coexist in equilibrium, provided that the endogenous borrowing limit is sufficiently large. Patient agents always hold money in such an equilibrium, but impatient ones never do. If the borrowing limit is positive, they would rather borrow at zero rates. If they do carry money, the borrowing limit collapses to zero.

The rest of the paper is structured as follows. Section 2 reviews the literature, Section 3 builds the model, Section 4 describes the constrained-efficient allocation, Section 5 discusses the existence of an equilibrium, Section 6 presents the main theoretical results, Section 7 presents some extensions to the main model and the last section concludes. Proofs are in the Appendix.

2 Literature review

The coexistence of money and credit has been investigated extensively in the theoretical monetary literature, but the types of credit studied vary across models. Specifically, the literature has focused on three categories of credit. First, there is real trade credit, which occurs when sellers of a good extend financing directly to the buyer in return for a payment of goods in latter periods. Second, there is financial credit, where buyers obtain a loan from a financial intermediary instead of the seller. In these models, financing is carried out in units of money—buyers borrow cash from the intermediary and repay in the same form. Finally, there is secured credit, whereby the buyer posts collateral to obtain a loan.

The paper by Gu, Mattesini, and Wright (2016) is a standard example of a model with trade credit. Sellers extend financing to buyers up to a limit and credit is an IOU promising repayment of goods in the next period. Their main result is that if the borrowing limit is low enough, then money and credit are perfect substitutes and changes in the credit limit are neutral. Lowering the credit limit just increases the demand for money and in turn increases its real value, which offsets any decrease in the credit limit. Consequently, credit is not essential in any monetary equilibrium. Other models of trade credit impose assumptions to ensure that money and credit are not perfect substitutes. Gomis-Porqueras, Peralta-Alva, and Waller (2014) assume that credit transactions can be monitored and, as a result, sellers must pay income taxes. Since cash trades cannot be monitored, there is a tax arbitrage opportunity for using cash. They show that for low inflation rates, credit is inessential. However, for sufficiently high inflation rates, buyers pay with cash for small transactions but use cash and credit for large transactions. In several papers, researchers have assumed that credit is not possible in all bilateral matches. Telyukova and Wright (2008) make such an assumption to explain why people hold money and have high credit card balances. Gomis-Porqueras and Sanches (2013) assume that credit transactions are only available in some bilateral matches and they are costly. As a result, money and credit coexist and both are essential. Lotz and Zhang (2016) assume sellers have to choose ex-ante whether to pay a fixed cost to access a credit system. If they do, then credit is available to buyers. They find that for a range of parameters money and credit coexist. Otherwise, either only money is used or only credit is used. Dong and Huangfu (2021), on the other hand, assume that the buyer can access a credit market by paying a fixed cost. They find equilibria similar to those in Lotz and Zhang. However, in the equilibrium where money and credit coexist, buyers are indifferent between using money or credit, much in the spirit of Gu, Mattesini, and Wright (2016). A critical point is that, in all of these models, credit is inessential at the Friedman rule.

Unlike in these models of trade credit, Berentsen, Camera, and Waller (2007) (henceforth BCW) developed a model of banking whereby all contracts involve transfers of money, not goods. Agents with idle cash can deposit it in a bank and earn interest, while those needing more cash can borrow it from the bank and pay interest. As was mentioned earlier, this financial credit arrangement improves welfare by compensating agents with idle cash. This makes money less costly to carry; so the demand for it increases, thereby raising its real value. Chiu and Meh (2011) and Boel and Camera (2020) amend BCW by introducing costly banking. Bencivenga and Camera (2011) study an environment similar to that in BCW but banks in their model are closer to the ones in Diamond and Dybvig (1983), which offer insurance contracts and undertake investment in physical capital. Agents withdrawing from the bank must pay a real transaction cost. They show that, away from the Friedman rule, banks improve welfare. Domínguez and Gomis-Porqueras (2019) assume buyers can acquire cash loans in some trades, but not in others. However, buyers also hold government bonds that can be liquidated to help finance consumption. Again, in all of these models, credit and banks are inessential at the Friedman rule.

Finally, there is a strand in the literature that studies the coexistence of secured credit and money. Ferraris and Watanabe (2008) and Ferraris and Watanabe (2011) assume agents can pledge their physical capital to banks to obtain cash loans to finance consumption. They then study how various monetary policies affect real allocations and the accumulation of capital. Ferraris and Mattesini (2020) use the Lagos and Wright (2005) framework to study a model where agents may randomly have two buying opportunities simultaneously—an opportunity to buy consumption goods and another to buy a valuable Lucas tree. Since agents need the money to buy goods, they cannot use it to buy the Lucas trees. However, the agent can pledge her current holdings of Lucas trees as collateral to acquire additional trees. Consequently, money and secured credit are both essential away from the Friedman rule. Yet, once again, at the Friedman rule credit is inessential—agents can carry enough money to buy the optimal amount of consumption and assets.

3 The model

The model builds on Berentsen, Camera, and Waller and Boel and Waller (2019). Time is discrete, the horizon is infinite, and there is a large population of infinitely lived agents who consume perishable goods and discount only across periods. In each period, agents may visit two sequential rounds of trade; we refer to the first as DM and the second as CM.

Rounds of trade differ in terms of economic activities and preferences. In the DM, agents face an idiosyncratic trading risk such that they either consume or produce. An agent consumes with probability α_b and produces with probability $1 - \alpha_b$. We refer to consumers as buyers and producers as sellers. Buyers get utility $u(q)$ from $q > 0$ consumption, where $u'(q) > 0$, $u''(q) < 0$, $u'(0) = +\infty$ and $u'(\infty) = 0$. Producers incur a utility cost y from supplying $y \geq 0$ labor to produce y goods. In the CM, everyone can consume and produce instead. As in Lagos and Wright (2005), agents have quasi-linear preferences $U(x) - n$, where the first term is utility from x consumption, and the second is disutility from n labor to produce n goods. We assume $U'(x) > 0$, $U''(x) \leq 0$, $U'(0) = +\infty$ and $U'(\infty) = 0$. Also, let q^* be the solution to $u'(q) = 1$ and x^* be the solution to $U'(x) = 1$.

The economy is subject to idiosyncratic demand shocks, with respect to which agents are heterogeneous. Specifically, at the beginning of each CM, agents draw an idiosyncratic time-preference shock $\beta_z \in \{\beta_L, \beta_H\}$ determining their interperiod discount factor. This implies that at the beginning of each period, an agent can be either patient (type H) with probability ρ or impatient (type L) with probability $1 - \rho$. We consider the case $0 < \beta_L < \beta_H < 1$ with no serial correlation in the draws. Note that time-preference shocks introduce ex-post heterogeneity across households, but the long-run distribution of time preferences is invariant.

3.1 Information frictions, money and credit

The preference structure we selected generates a single-coincidence problem in the DM, since buyers do not have a good desired by sellers. Moreover, two additional frictions characterize the DM. First, agents are anonymous as in Kocherlakota (1998), since trading histories of agents in the goods markets are private information. This in turn rules out trade credit between individual buyers and sellers. Second, there is no public communication of individual trading outcomes, which in turn eliminates the use of social punishments to support gift-giving equilibria. The combination of these two frictions together with the single-coincidence problem implies that sellers require immediate compensation from buyers. So, buyers must use money to acquire goods in the DM.

Agents can borrow cash from a bank to supplement their money holdings, but do so at the cost of the nominal interest rate. Following BCW, we assume banks can do so because they operate a record-keeping technology of financial histories (but not trading histories) at zero cost. Note that since record keeping can only be done for financial transactions, trade credit between buyers and sellers is not feasible. This implies that money is still essential to trade in the DM even if credit is available via financial intermediaries. We also assume that all financial contracts are one-period contracts, which are optimal in this economy due to quasi-linear preferences.

Money is not essential for trade in the CM, and agents can finance their consumption by getting credit, working, or using money balances acquired earlier. To model CM credit, we assume agents are allowed to borrow and lend through selling and buying nominal bonds, subject to an exogenous credit constraint A. Specifically, agents lend $-p_{at}a_{t+1}$ (or borrow $p_{at}a_{t+1}$), where p_{at} is the price of a bond that delivers one unit of money in $t+1$, and receive back a_t (or pay back $-a_t$). We assume that any funds borrowed or lent in the CM are repaid in the following CM. One can show that, even with quasi-linearity of preferences in the CM, there are gains from multi-period contracts due to time-preference shocks.

Of course, default is a serious issue in all models with credit. However, we simplify the analysis by assuming a mechanism exists that ensures repayment of loans in the CM. In the DM, as a benchmark we first investigate the case in which banks can force repayment of loans at no cost. As an extension, we then consider the case in which banks cannot force repayment of loans and therefore borrowers have an incentive to default. The penalty for this will be permanent exclusion from the banking sector, and for credit to exist we will need to ensure voluntary repayment.

3.2 Policy tools

We assume there is a government that is in charge of monetary policy and is the only supplier of fiat money, of which there is an initial stock $M_0 > 0$. We denote the gross growth rate of money supply by $\pi = M_t/M_{t-1}$, where M_t denotes the money stock in the CM in period t. The central bank implements its long-term inflation goal by providing deterministic lump-sum injections of money $\tau = (\pi - 1)M_{t-1}$, which are distributed to private agents at the beginning of the CM. If $\pi > 1$, agents receive lump-sum transfers of money, whereas for $\pi < 1$ the central bank must be able to extract money via lump-sum taxes from the economy.

4 Constrained-efficient allocation

We start by discussing the allocation selected by a benevolent planner subject to the same physical and informational constraints faced by the agents. We will refer to this allocation as constrained-efficient. The environment's frictions imply that the planner can observe neither types nor identities in the DM and therefore has no ability to transfer resources across agents over time in that market. Furthermore, at the start of the DM, all agents are identical ex-ante, since the previous period's β shock is no longer relevant and the DM shocks have not been realized. Thus, if we look at welfare from this point in time, we effectively have a representative agent problem.

Therefore, the planning problem in the DM corresponds to a sequence of static maximization problems subject to the technological constraints. This implies that in the DM the planner must solve:

$$\begin{aligned} \underset{q,y}{Max} \ \alpha_b \ u(q) - (1-\alpha_b)y \\ \text{s.t. } \alpha_b q = (1-\alpha_b)y \end{aligned} \tag{1}$$

In the CM, once the discount-factor shocks are realized, the agents are heterogeneous with regard to intertemporal choices. We also do not have the informational frictions in this market that exist in the DM. Consequently, the planner can transfer resources across agents over time and therefore chooses consumption and labor sequences $\{x_{j0}, x_{j1}, ..\}$ and $\{n_{j0}, n_{j1}, ..\}$ for $j = H, L$ that maximize a weighted sum of individual utility functions subject to feasibility and non-negativity constraints:

$$\begin{aligned} Max \sum_{j=H,L} \sigma_j \left[U(x_{j0}) - n_{j0} + \sum_{t=1}^{\infty} \beta_j \beta^{t-1} (U(x_{jt}) - n_{jt}) \right] \\ \text{s.t. } \rho x_{Ht} + (1-\rho) x_{Lt} = \rho n_{Ht} + (1-\rho) n_{Lt} \quad \text{for } t = 0,1,2,... \\ \text{s.t. } n_{jt} \geq 0 \quad \text{for } j = H, L \text{ and } t = 0,1,2,... \end{aligned} \tag{2}$$

where $\beta = \rho\beta_H + (1-\rho)\beta_L$ is the average discount fator and σ_H and σ_L are positive utility weights. A solution to this problem is characterized by:

$$U'(x_{j0}) = 1 - \mu_t^j / \sigma_j \qquad \text{for } j = H, L \text{ and } t = 0 \tag{3}$$

$$U'(x_{jt}) = 1 - \mu_t^j / \sigma_j \beta_j \beta^{t-1} \qquad \text{for } j = H, L \text{ and } t \geq 1 \tag{4}$$

where μ_t^j denotes the Kuhn-Tucker multiplier associated with the non-negativity constraint on n_{jt}. Note that the difference between equations (3) and (4) implies a different allocation when $t = 0$ than when $t \geq 1$. In short, once $t = 1$ is reached, the planner would prefer to reoptimize and give each agent the allocation solving (3) rather than (4) evaluated at $t = 1$. Therefore, the social planner's problem is not time consistent.[2] Consequently, satisfying (3) and (4) requires that the planner be able to commit to future promises of consumption and labor in the CM exchange. If the planner cannot commit to fulfill such promises, then the only consistent solution to this problem is $\mu_t^j = 0$ in all periods. This implies that a discretionary planner's allocation has $U'(x_{jt}) = 1$ and $n_{jt} > 0$ for $j = H, L$ and $t \geq 0$—the discretionary planner wants both types to work and consume a constant and equal amount

2. See also the discussion in Drugeon and Wigniolle (2016).

in every period. We adopt the allocation corresponding to the discretionary planner as our benchmark for welfare. We do so for several reasons. First, at the beginning of the DM all agents are ex-ante identical. So, viewing welfare from this point in time is equivalent to having a representative agent problem. Second, there are no ex-post welfare gains from transferring labor across agents based on the discount-factor shocks because of quasi-linear utility—shifting labor from one agent to fulfill earlier promises is zero-sum ex-post.

In sum, in the constrained-efficient allocation we focus on marginal consumption utility equals marginal production disutility in each market and in every period. Such an allocation is therefore stationary and defined by $u'(q) = 1$ in the DM and $U'(x) = 1$ in the CM. The constrained-efficient consumption is therefore defined uniquely by $q_H = q_L = q^*$ and $x_H = x_L = x^*$, thus implying equal consumption for type H and type L agents.

5 Stationary monetary allocations

In what follows, we want to determine if the constrained-efficient allocation can be decentralized in a monetary economy with competitive markets. Thus, we focus on stationary monetary outcomes such that end-of-period balances of real money and bonds are time invariant.

We simplify notation omitting t subscripts and use a prime superscript $'$ and a -1 subscript to denote variables corresponding to the next and previous period, respectively. We let p_1 and p_2 denote the nominal price of goods in the DM and the CM, respectively, of an arbitrary period t. We also let β_j and β_z denote the discount factors drawn in period $t-1$ and t, respectively. In addition, we normalize all nominal variables by p_2, so that DM trades occur at the real price $p = p_1/p_2$. In this manner, the timing of events in any period t can be described as follows.

An arbitrary agent of type $j = H, L$ enters the DM in period t with a portfolio $\omega_j = (m_j, a_j)$ listing $m_j = m(\beta_j)$ real money holdings and $a_j = a(\beta_j)$ bonds from the preceding period after experiencing a time-preference shock β_j. Trading shocks $k = b, s$ are then realized, where b and s identify a buyer and a seller respectively. The banking sector then opens and the agent decides if and how much she wants to borrow $\ell_j^k = \ell(\beta_j)$ and/or deposit $d_j^k = d(\beta_j)$, where $k = b, s$ denotes the trading shock experienced in the DM. It is straightforward to show that $d_j^b = \ell_j^s = 0$. Thus, we use the notation $d_j^s = d_j$ and $\ell_j^b = \ell_j$. Finally, the banking sector closes and agents trade goods. Note that the bank closes before the onset of trading in the DM, which implies that sellers cannot deposit receipts of cash earned from selling in the DM.

After the DM closes, an agent of type j enters the CM with portfolio $\omega_j^k = (m_j^k, a_j)$, where $m_j^k = m_j^k(\beta_j)$ denotes money holdings carried over from the DM. If we let $q_j = q(\beta_j)$ denote consumption and $y_j = y(\beta_j)$ production in the DM, individual real money holdings for an agent j evolve as follows:

$$\begin{aligned} m_j^b &= m_j + \ell_j - pq_j \\ m_j^s &= m_j - d_j + py_j \end{aligned} \tag{5}$$

That is, buyers borrow but deplete balances by pq_j while sellers deposit and increase balances by py_j. Idiosyncratic time-preference shocks β_z are realized at the beginning of the CM. Left-over cash is then used to trade and settle bonds positions and x and n are, respectively, consumption bought and production sold in the CM. Note that bond positions a_j at the beginning of the CM are not affected by trading shocks in the DM, since bonds can only be used in the CM. Agents repay loans plus interest $(1 + i)\ell_j$ if they were borrowers in the DM, get $(1 + i_d)d_j$ if they were depositors, and receive lump-sum transfers τ. They also adjust their money balances $m'_z = m'(\beta_z)$ and decide whether they want to borrow or lend $a'_z = a'(\beta_z)$, where m'_z and a'_z denote real values of money holdings and bonds at the start of tomorrow's DM. Figure 1 displays the timeline of shocks and decisions within each period:

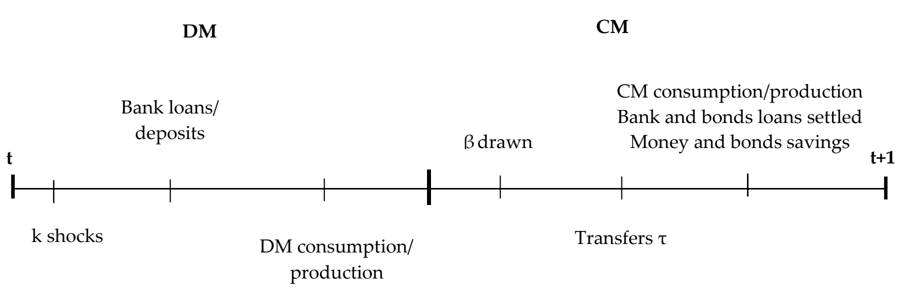

Figure 1: Timing of events within a period.

Since we focus on stationary equilibria where end-of-period real money balances are time and state invariant so that $M/p_2 = M'/p'_2$, we have that:

$$\frac{p'_2}{p_2} = \frac{M'}{M} = \pi \tag{6}$$

which implies that the inflation rate equals the growth rate of the money supply. The government budget constraint therefore is:

$$\tau = (\pi - 1)[\rho m_H + (1 - \rho)m_L] \tag{7}$$

5.1 The CM problem

Given the recursive nature of the problem, we use dynamic programming to analyze the problem of an agent j at any date, with $j = H, L$. We let $V(\omega_j)$ denote the expected lifetime utility for an agent entering the DM with portfolio ω_j before trading shocks are realized. We also let $W_z(\omega_j^k)$ denote the expected lifetime utility from entering the CM with portfolio ω_j^k and receiving a discount factor shock β_z at the beginning of the CM. The agent's problem at the start of the CM then is:

$$\begin{aligned} W_z(\omega_j^k) = & \underset{x_{jz}^k, n_{jz}^k, a_z', m_z'}{Max} \quad U(x_{jz}^k) - n_{jz}^k + \beta_z V(\omega_z') \\ \text{s.t.} \; & x_{jz}^k + a_j + (1+i)\ell_j + \pi m_z' = n_{jz}^k + m_j^k + (1+i_d)d_j + \tau + p_a \pi a_z' \\ & a_z' \leq A \\ & m_z' \geq 0 \end{aligned} \tag{8}$$

where $A \geq 0$ is the borrowing constraint for CM bonds. Also, i and i_d are the nominal interest rates paid on a bank loan and received on a bank deposit respectively. The budget constraint in the problem above expresses the idea that resources available in the CM to an agent j depend on the realization of the DM trading shock k, as well as the idiosyncratic shocks β_j and β_z. Specifically, an agent has m_j^k real balances carried over from the DM and is able to borrow $\pi a_z'$ (or lend if $a_z' < 0$) at a price p_a. Other resources are n_{jz}^k receipts from current sales of goods, lump-sum transfers τ, and deposits plus interest $(1 + i_d)d_j$. These resources can be used to finance current consumption x_{jz}^k, to pay back loans a_j and $(1+i)\ell_j$, and to carry $\pi m_z'$ real money balances into the next period. The factor $\pi = p_2'/p_2$ multiplies a_z' and m_z' because the budget constraint is expressed in real terms and both money and bonds are nominal assets. Conditions for $n_{jz}^k \geq 0$ are in Lemma 3.

Rewriting the constraint in terms of n_{jz}^k and substituting into (8), we find that in a stationary monetary economy we must have:

$$1 = \frac{\partial W_z(\omega_j^k)}{\partial m_j^k} = -\frac{\partial W_z(\omega_j^k)}{\partial a_j} \tag{9}$$

This result depends on the quasi-linearity of the CM preferences and the use of competitive pricing. It implies that the marginal valuation of real balances and bonds in the CM is identical and does not depend on the agent's current type z or past type j, wealth ω_j^k, or trade shock k. Moreover:

$$\frac{\partial W_z(\omega_j^k)}{\partial \ell_j} = -(1+i) \text{ and } \frac{\partial W_z(\omega_j^k)}{\partial d_j} = 1 + i_d \tag{10}$$

This allows us to disentangle the agents' portfolio choices from their trading histories since:

$$W_z(\omega_j^k) = W_z(0) + m_j^k - a_j - (1+i)\ell_j + (1+i_d)d_j$$

i.e., the agent's expected value from having a portfolio ω_j^k at the start of a CM is the expected value from having no wealth, $W_z(0)$, letting $\omega_j = (0,0) \equiv 0$, plus the current real value of net wealth $m_j^k - a_j - (1+i)\ell_j + (1+i_d)d_j$. Note also that everyone consumes identically in the CM since:

$$U'(x_{jz}^k) = 1 \tag{11}$$

which also implies $x_{jz}^k = x^*$. That is, everyone consumes x^* independent of current type and past shocks, the reason being that agents in the CM can produce any amount at a constant marginal cost. Thus, goods market clearing in the CM requires:

$$x^* = \alpha_b N^b + (1-\alpha_b)N^s \tag{12}$$

where $N^k = \rho^2 n_{HH}^k + \rho(1-\rho)(n_{LH}^k + n_{HL}^k) + (1-\rho)^2 n_{LL}^k$ is labor effort for all agents who experienced a trading shock k in the DM. Let $\mu_z^m \geq 0$ denote the Kuhn-Tucker multiplier associated with the non-negativity constraint for money. Also, let λ_z^a denote the multiplier on the CM borrowing constraint. The first-order conditions for the optimal portfolio choice are:

$$1 = \frac{\beta_z}{\pi}\frac{\partial V(\omega_z')}{\partial m_z'} + \frac{\mu_z^m}{\pi} \tag{13}$$

$$-p_a = \frac{\beta_z}{\pi}\frac{\partial V(\omega_z')}{\partial a_z'} - \frac{\lambda_z^a}{\pi} \tag{14}$$

The left-hand sides of the expressions above define the marginal cost of the assets. The right-hand sides define the expected marginal benefit from holding the asset, either money or bonds, discounted according to time preferences and inflation. From (13) and (14) we know that money holdings m_z' and bonds a_z' are independent of trading histories and past demand shocks, but instead depend on the current type z and the expected marginal benefit of holding money and bonds in the DM, which may differ across types. We will study this next.

5.2 The DM problem

An agent with a portfolio ω_j at the opening of the DM before trading shocks are realized has expected lifetime utility:

$$V(\omega_j) = \alpha_b V^b(\omega_j) + (1-\alpha_b)V^s(\omega_j) \tag{15}$$

First, we determine y_j. The seller's problem depends on the current disutility of production and the expected continuation value. Specifically, the seller's problem can be written as:

$$\begin{aligned} V^s(\omega_j) = \underset{y_j, d_j}{Max} & \; -y_j + \rho W_H(\omega_j^s) + (1-\rho)W_L(\omega_j^s) \\ \text{s.t. } & \; d_j \leq m_j \end{aligned} \tag{16}$$

where the constraint means that deposits must be financed by money holdings. The first-order conditions, together with (5) and (9), give:

$$p = 1 \tag{17}$$

$$i_d = \lambda_j^s \tag{18}$$

where λ_j^s is the multiplier on the deposit constraint and (17) implies that production is not type dependent, i.e. $y_j = y$ for $j = H, L$. Moreover, if $i_d > 0$, then it must be that $\lambda_j^s > 0$, and therefore, a seller of type j will deposit all her money holdings, so that $d_j = m_j$.

Now, we determine q_j. A buyer's problem is:

$$\begin{aligned} V^b(\omega_j) = \underset{q_j, \ell_j}{Max} & \; u(q_j) + \rho W_H(\omega_j^b) + (1-\rho)W_L(\omega_j^b) \\ \text{s.t. } & \; pq_j \leq m_j + \ell_j \quad \text{and} \quad \ell_j \leq \bar{\ell} \end{aligned} \tag{19}$$

The first constraint means consumption can be financed by money holdings and bank loans. The second constraint exists because borrowers are subject to a technological constraint, $\bar{\ell}$, that depends on the funds available to the bank and is determined in equilibrium. Using (5), (9), and (17), the first-order condition with respect to q_j implies:

$$u'(q_j) = 1 + \lambda_j^b \tag{20}$$

where λ_j^b is the Lagrange multiplier on the buyer's budget constraint. From (17) and (20) we know that if the buyer is constrained and $\lambda_j^b > 0$, then $u'(q_j) > 1$ and $q_j < q^*$. If instead the buyer is unconstrained and therefore $\lambda_j^b = 0$, then $u'(q_j) = 1$ and $q_j = q^*$. The first-order condition with respect to ℓ_j instead implies:

$$i = \lambda_j^b - \lambda_j^\ell \tag{21}$$

where λ_j^ℓ is the Lagrange multiplier on the buyer's borrowing constraint. If $i > 0$, it must be that $\lambda_j^b > 0$, and therefore, a buyer will be constrained in the DM and $q_j < q^*$. If instead $i = 0$, two cases are possible: (i) $\lambda_j^b = \lambda_j^\ell = 0$ so that the borrowing constraint does not bind, buyers are unconstrained, and $q_j = q^*$; and (ii) $\lambda_j^b = \lambda_j^\ell > 0$ so that the borrowing constraint binds, buyers are constrained, and $q_j < q^*$.

Goods market clearing requires:

$$(1 - \alpha_b)y = \alpha_b[\rho q_H + (1 - \rho)q_L] \tag{22}$$

The bond market clearing condition instead is such that:

$$\rho a_H + (1 - \rho)a_L = 0 \tag{23}$$

5.3 Bank's problem

Banks accept nominal deposits D on which they pay the nominal interest rate i_d, and make nominal loans $\mathcal{L}$ at the nominal interest rate i. The banking sector is perfectly competitive with free entry, so banks take these rates as given. There is no strategic interaction among banks or between banks and agents and there is no bargaining over the terms of the loan contract. We also assume repayment of bank loans can be enforced at no cost, there are no reserve requirements, and banks face no operational costs. There is, however, a feasibility constraint limiting the amount of loans to be no more than the amount of deposits taken in, so that $\mathcal{L} \leq D$. Banks cannot recognize agents' types and therefore will charge the same interest rate i and impose the same borrowing limit on all agents. The representative bank therefore solves the following problem in every period:

$$\underset{\mathcal{L},D}{Max}\ i\mathcal{L} - i_d D \tag{24}$$

subject to the balance-sheet constraint:

$$\mathcal{L} \leq D$$

where $D = (1-\alpha_b)[\rho d_H + (1-\rho)d_L]$ is total deposits and $\mathcal{L} = \alpha_b[\rho\ell_H + (1-\rho)\ell_L]$ is total loans. Market clearing in the banking sector implies:

$$(1-\alpha_b)[\rho d_H + (1-\rho)d_L] = \alpha_b[\rho\ell_H + (1-\rho)\ell_L] \tag{25}$$

With free entry the bank makes zero profits, and from (25) we have that:

$$i = i_d \tag{26}$$

5.4 Marginal value of money

To find optimal savings for an agent j use (8), (15), (16), (17), and (19) to obtain:

$$V(\omega_j) = m_j - a_j + \alpha_b[u(q_j) - q_j] + i_d(1-\alpha_b)d_j - i\alpha_b\ell_j + \rho W_H(0) + (1-\rho)W_L(0) \tag{27}$$
$$\text{s.t. } a'_z \leq A,\ \ m'_z \geq 0,\ \ q_j \leq m_j + \ell_j,\ \ \ell_j \leq \bar{\ell},\ \ d_j \leq m_j$$

The expected lifetime utility $V(\omega_j)$ therefore depends on the agent's net wealth $m_j - a_j$ and three other elements. First, there is the expected surplus from trade in the DM—with probability α_b the agent spends q_j on consumption deriving utility $u(q_j)$. Second, there is the expected intermediation rent $i_d(1-\alpha_b)d_j - i\alpha_b\ell_j$. Third, there is the expected continuation payoff $\rho W_H(0) + (1-\rho)W_L(0)$. Therefore, the marginal value of money satisfies:

$$\frac{\partial V(\omega_j)}{\partial m_j} = \alpha_b u'(q_j) + (1-\alpha_b)(1+i_d) \tag{28}$$

If the agent is a buyer, she receives the marginal benefit $u'(q_j)$ from using money to finance consumption. If instead she is a seller, she receives $(1+i_d)$ from depositing money in the bank. The marginal value of bonds instead satisfies:

$$\frac{\partial V(\omega_j)}{\partial a_j} = -1 \tag{29}$$

Note that (28) and (29) imply that money is valued dissimilarly by agents, whereas bonds are valued identically in the economy. Combining (13) with (28) and (14) with (29) one gets that in equilibrium the following Euler equations must hold:

$$\frac{\pi - \beta_j}{\beta_j} = \alpha_b[u'(q_j) - 1] + (1-\alpha_b)i_d + \frac{\mu_j^m}{\beta_j} \tag{30}$$

and

$$\pi p_a = \beta_j + \lambda_j^a \tag{31}$$

We now want to investigate whether a CM to CM bond would indeed circulate in this economy. As in Boel and Waller (2019), we find that the following result holds:

Lemma 1. *A stationary monetary equilibrium exists with impatient agents borrowing and patient agents lending at a price* $p_a = \beta_H/\pi$. *Specifically,* $a_L = A$ *and* $a_H = -(1-\rho)A/\rho$.

That is, impatient agents borrow and patient ones lend in order to smooth the labor effort. Once we know the price at which these bonds circulate in equilibrium, we can pin down their net nominal yield, which is:

$$i_a = \frac{1}{p_a} - 1 \quad \Rightarrow \quad i_a = \frac{\pi}{\beta_H} - 1 \tag{32}$$

Note that there are two nominal interest rates in our model. The first one is i_a, which is the rate on an illiquid asset and is affected directly by long-term monetary policy through π and the discount rate of the impatiet agents $1/\beta_H$. The second nominal interest rate in the model is the bank's rate $i = i_d$. Combining (20), (21), (26) and (32) we have that (30) for $j = H, L$ can be written as:

$$\frac{\pi - \beta_j}{\beta_j} = i + \alpha_b \lambda_j^\ell + \frac{\mu_j^m}{\beta_j} \tag{33}$$

$$i_a = i + \alpha_b \lambda_H^\ell + \frac{\mu_H^m}{\beta_H} \tag{34}$$

The expression in (34) illustrates a no-arbitrage condition between illiquid bonds and deposits. When an agent holds an additional unit of money, she gives up the interest rate i_a but earns the rate i. Note also that, given (33) and (34), long-term monetary policy also controls the bank rate i via changes in the growth rate of the money supply π. The following result holds:

Lemma 2. *Any stationary monetary equilibrium must be such that* $\pi \geq \beta_H$, *i.e.* $i_a \geq 0$.

This result derives from a simple no-arbitrage condition—in a monetary equilibrium, the value of money cannot grow too fast with $\pi < \beta_H$ or else type H agents will not spend it.[3] This, together with (32), implies that to run the Friedman rule the monetary authority must let $\pi \to^+ \beta_H$ and cannot target β_L instead.

We must also ensure that the labor effort in the CM is non-negative and this is guaranteed by the condition in Lemma 3.

3. The result that the rate of return on the asset cannot exceed the lowest rate of time preference is common to other models with heterogeneous time preferences. See, for example, Becker (1980), Boel and Camera (2006), and Boel and Waller (2019).

Lemma 3. *If $i_a = 0$, $A \leq [x^* - (1 + \rho\beta_H)q^*]/[\beta_H + (1 - \rho)/\rho]$ guarantees that $n^k_{jz} \geq 0$ in a stationary monetary equilibrium.*

The intuition is that agents work if the debt limit is tight enough, and how tight depends on the difference between x^* and q^*.

6 Equilibrium with perfect enforcement

In this section, as a benchmark, we assume that banks can force repayment of loans at no cost. We want to study the coexistence of money and credit at $i_a = 0$, so we will focus on the case $\pi \to^+ \beta_H$. Throughout, we will be making the limit argument that agents still want to deposit if interest rates are approaching zero. We can then state the following:

Definition 1. *A symmetric stationary monetary equilibrium consists of m_j and a_j satisfying* (30) *and* (31) *for $j = H, L$.*

The reason is that once the equilibrium stocks of money and bonds are determined, all other endogenous variables can be derived. The following result holds:

Proposition 1. *When $i_a = 0$, then $m_H > 0$ and $q_H = q^*$. Three possible cases exist for type L agents: (i) $m_L > 0$ and $q_L = \tilde{q} < q^*$ if $\bar{\ell} \leq \tilde{q}$, where $\tilde{q}$ solves* (30) *for $j = L$ and $\mu^m_L = 0$; (ii) $m_L = 0$ and $\tilde{q} < q_L < q^*$ if $\tilde{q} < \bar{\ell} < q^*$; and (iii) $m_L = 0$ and $q_L = q^*$ if $\bar{\ell} \geq q^*$.*

Proposition 1 has several implications. First, patient agents always bring money when $i_a = 0$. The intuition is that money is costless to carry across periods for patient agents when rates are zero and so they bring enough to afford the efficient allocation in the DM. Money is costly for impatient agents, since $\pi > \beta_L$ at $i_a = 0$ and, therefore, the shadow interest rate is positive for them. This implies that impatient agents will bring money only if they are sufficiently constrained, that is, if the borrowing limit is tight enough at $\bar{\ell} \leq \tilde{q}$. In this case, money and credit are perfect substitutes and L agents always consume $\tilde{q}$. Thus, tightening the debt limit in this case is neutral, much like in Gu, Mattesini, and Wright 2016. As the borrowing limit starts increasing, L agents prefer not to carry money, since borrowing is costless when $i_a = i = 0$. In this case, increasing the debt limit improves the allocation. Indeed, if there are sufficient deposits so that $\bar{\ell} \geq q^*$, the banking sector can fix the problems highlighted in Lemma 2 and both patient and impatient agents are able to achieve the efficient allocation. See Figure 2.

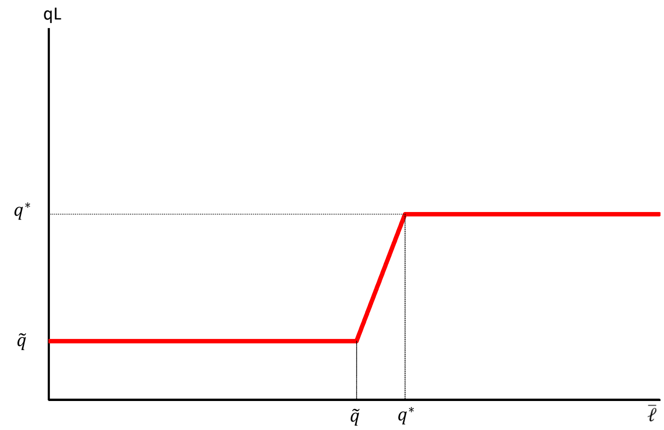

Figure 2: Consumption of type L agents as a function of the debt limit. The horizontal axis reports the exogenous debt limit $\bar{\ell}$. The vertical axis reports consumption of type L agents q_L. For $\bar{\ell} \leq \tilde{q}$, L agents always consume $\tilde{q}$ and are indifferent between money and credit. For $\tilde{q} < \bar{\ell} < q^*$, L agents do not bring money but are still constrained, but they are unconstrained if $\bar{\ell} \geq q^*$. The figure is drawn with the utility function $u(q) = ((q+b)^{1-\delta} - b^{1-\delta})/(1-\delta)$, discount factors $\beta_H = 0.98$ and $\beta_L = 0.95$, a measure of buyers $\alpha_b = 0.2$, and a measure of type H agents $\rho = 0.3$, $\delta = 0.99999$ and $b = 0.00001$.

From Proposition 1, we know that consumption for type L agents with financial intermediation is higher if $\bar{\ell} > \tilde{q}$ than with only money. This implies that financial intermediation can improve the allocation even at zero interest rates when agents have different time preferences. See Figure 3.

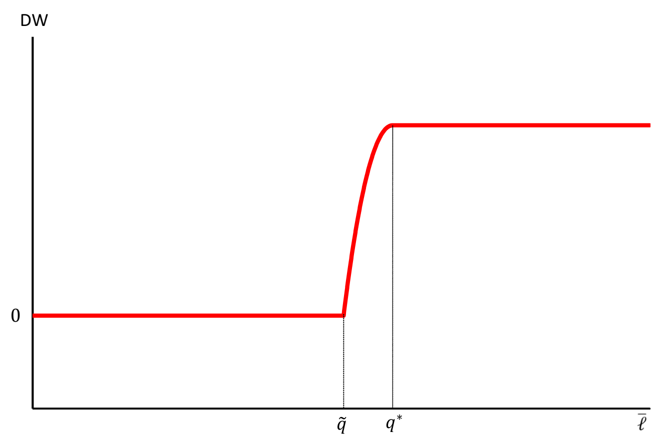

Figure 3: Difference in welfare for type L agents with and without financial intermediation. The horizontal axis reports the exogenous debt limit $\bar{\ell}$. The vertical axis reports the measure DW, which is the difference in ex-ante welfare levels with and without banks. The difference is equal to zero for $\bar{\ell} \leq \tilde{q}$ and it is instead positive for $\bar{\ell} > \tilde{q}$. The figure is drawn with the utility function $u(q) = ((q+b)^{1-\delta} - b^{1-\delta})/(1-\delta)$, discount factors $\beta_H = 0.98$ and $\beta_L = 0.95$, a measure of buyers $\alpha_b = 0.2$, and a measure of type H agents $\rho = 0.3$, $\delta = 0.99999$ and $b = 0.00001$.

But why does banking improve the allocation when rates are zero? In this case, banking does not pay any interest to depositors and so its welfare-increasing role must be due to its ability to relax borrowers' liquidity constraints.

Corollary 1. *Let $i_a = 0$. For $\bar{\ell} > \tilde{q}$, the gain in welfare from financial intermediation comes from the relaxation of borrowers' liquidity constraints and not from the payment of interest to depositors.*

7 Extensions

The results on stationary monetary allocations in Section 6 are robust to several types of extensions, as explained in the following subsections.

7.1 Costly banking

Imagine that agents have to incur a fixed effort/utility cost η to borrow, but no cost to deposit. Then, (27) becomes:

$$V^b(\omega_j^b) = m_j^b - a_j + \alpha_b[u(q_j) - q_j] - \iota(\ell_j)\eta + \bar{W}(0) \quad (35)$$
$$\text{s.t. } a_z' \leq A, \;\; m_z' \geq 0, \;\; q_j \leq m_j + \ell_j, \;\; \ell_j \leq \bar{\ell}, \;\; d_j \leq m_j$$

where $\bar{W}(0) = \rho W_H(0) + (1-\rho)W_L(0)$ and the indicator function $\iota(\ell_j)$ is:

$$\iota(\ell_j) = \begin{cases} 1 & \text{if} \quad \ell_j > 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that type H agents hold enough money to buy q^* and so they will never pay the fixed cost η. A buyer of type L will pay the fixed cost η only if that entails a higher welfare. Given that without banking the agent will be able to afford $\tilde{q}$ defined in (30), the condition for being willing to pay the fixed cost is:

$$u(q_L) - q_L - \eta \geq u(\tilde{q}) - \tilde{q}$$

and therefore:

$$\frac{u(q_L) - u(\tilde{q})}{q_L - \tilde{q}} \geq 1 + \frac{\eta}{q_L - \tilde{q}}$$

meaning that an agent L is willing to pay the fixed cost η only if she is sufficiently constrained, in which case the Euler equation is the same as in (30). That is, banking can still be welfare increasing even if costly provided that the debt limit is high enough and the fixed cost is not too high. See Figure 4.

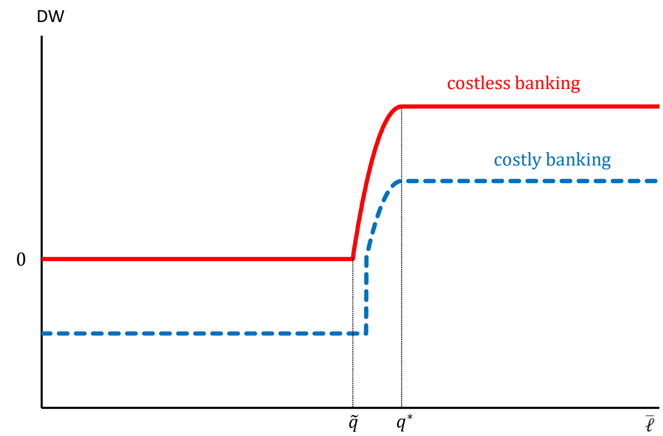

Figure 4: Difference in welfare for type L agents with and without costly financial intermediation. The horizontal axis reports the exogenous debt limit $\bar{\ell}$. The vertical axis reports the measure DW, which is the difference in ex-ante welfare levels with and without banks. The solid line is DW when banking is costless, whereas the dashed line is DW when borrowing entails the fixed utility cost η. The figure is drawn with the utility function $u(q) = ((q+b)^{1-\delta} - b^{1-\delta})/(1-\delta)$, discount factors $\beta_H = 0.98$ and $\beta_L = 0.95$, a measure of buyers $\alpha_b = 0.2$, and a measure of type H agents $\rho = 0.3$, $\delta = 0.99999$, $b = 0.00001$ and $\eta = 0.005$.

7.2 Aggregate demand shocks

We now investigate whether the coexistence of money and credit at zero interest rates outlined in Proposition 1 holds for any type of heterogeneity or if instead it hinges specifically on agents having different time preferences. Assume, for example, that agents receive preference shocks in the DM in the form of marginal utility shocks instead of discount factor ones in the CM. That is, in each period in the DM agents have preferences $\epsilon_\gamma u(q_\gamma)$ for $\gamma = 1, 2$ with $\epsilon_1 > \epsilon_2$ and type 1 and type 2 agents in proportions ζ and $1-\zeta$, respectively. In this case, the Euler equation for money (30) becomes:

$$\frac{\pi - \beta}{\beta} = \alpha_b[\epsilon_\gamma u'(q_\gamma) - 1] + (1-\alpha_b)i + \frac{\mu_\gamma^m}{\beta} \tag{36}$$

From (36) we see that $\pi \to^+ \beta$ implies $i = 0$ and $\mu_\gamma^m = 0$ for $\gamma = 1, 2$. Thus, both type 1 and 2 agents bring money at zero interest rates. Morevoer, from (36) we know that $\epsilon_\gamma u'(q_\gamma) = 1$ when interest rates are zero, so that both type 1 and 2 agents are unconstrained and can afford the efficient quantity q^* at $i = 0$.[4] How does this allocation differ from one without

4. Note also that in this case there is no CM borrowing and lending, since agents have the same discount factor. In this case, $i_a = (\pi - \beta)/\beta$ should be interpreted as the interest rate on an illiquid bond that does not circulate but can still be priced.

financial intermediation? In that case, (30) becomes

$$\frac{\pi - \beta}{\beta} = \alpha_b[\epsilon_\gamma u'(q_\gamma) - 1] \tag{37}$$

The key result is that in this case financial intermediation does not improve the allocation at zero interest rates since we still have $q_\gamma = q^*$ for $i = 0$. The reason is that agents can perfectly self-insure against consumption risk only with money because the shadow rate is zero for both types 1 and 2. It is straightforward to show that this result goes through for several other types of heterogeneity as well, as long as agents have the same discount factor.

7.3 Endogenous bank debt limit

We now focus on an equilibrium where banks cannot force repayment of loans, thus implying that borrowers might have an incentive to default. We assume agents make their default decisions after time-preference shocks are realized. Note that we focus solely on the choice of agents who were buyers in the DM because they are the only ones with loans to repay and thus have a possible incentive to renege on their debts. The short-term benefit from defaulting is additional leisure, as the agent will not have to work to repay her loan. The long-term cost is exclusion from the banking system in the following periods. The representative bank in this case solves the following problem:

$$\begin{aligned} &\underset{\ell}{Max}\ i\ell - i_d D \\ &\text{s.t. } \ \ell \leq \bar{\ell} \\ &\qquad u(q_L) - (1+i)\ell \geq \Gamma \end{aligned} \tag{38}$$

where ℓ is loans from type L agents and Γ is the reservation value of a borrower of type L, i.e., the borrower's surplus from receiving a loan at another bank. With free entry the bank makes zero profits and therefore $i = i_d$. The first-order condition for the bank's problem then becomes:

$$-\lambda_L^B + \lambda_L^\Gamma \left[u'(q_L)\frac{dq_L}{d\ell} - (1+i) \right] = 0$$

where λ_L^B and λ_L^Γ are the Lagrange multipliers on the bank's technology constraint and on the borrower's participation constraint, respectively. From (17) and rearranging, we get:

$$u'(q_L) = 1 + i + \lambda_L^B / \lambda_L^\Gamma \tag{39}$$

Note that banks will always choose a loan size such that $\lambda_L^\Gamma > 0$. Then, if $\lambda_L^B > 0$, the constraint on the loan size is binding. This implies $u'(q_L) > 1$ and therefore $q_L < q^*$ from (20).

In order for a monetary equilibrium with banking to exist, we must ensure voluntary repayment, i.e., agents prefer not to default, thus implying $W_z(\omega_j^b) \geq \hat{W}_z(\omega_j^b)$, where the hat indicates the optimal choice by a deviator. The endogenous borrowing constraint therefore must satisfy:

$$W_z(\omega_j^b) = \hat{W}_z(\omega_j^b) \tag{40}$$

where $W_z(\omega_j^b)$ and $\hat{W}_z(\omega_j^b)$ denote the expected discounted utility for buyers entering the CM and repaying their loans or defaulting on their loans, respectively, so that:

$$W_z(\omega_j^b) = U(x^*) - n_{jz}^b + \beta_z V(\omega_z') \tag{41}$$

and

$$\hat{W}_z(\omega_j^b) = U(\hat{x}_{jz}^b) - \hat{n}_{jz}^b + \beta_z \hat{V}(\hat{\omega}_z') \tag{42}$$

We can now state the following:

Definition 2. *With an endogenous debt limit, a symmetric stationary monetary equilibrium consists of a m_j and a_j satisfying* (30), (31) *and* (40).

In this economy, the growth rate of the money supply π affects not only the marginal value of money, but also the value of either staying in the banking system $W_z(\omega_j^b)$ or defaulting $\hat{W}_z(\omega_j^b)$. This imposes constraints on the inflation rate π the monetary authority can impose in a monetary equilibrium while still having a functioning banking system. We will investigate such constraints next, and in particular, we want to understand if money and banking can coexist when interest rates are zero and borrowers can default on their loans. We find the following result holds:

Proposition 2. *Let $i_a = 0$. There exists a monetary equilibrium with $m_H > m_L = 0$ and an endogenous borrowing limit $\bar{\ell} > 0$.*

From the proof of Proposition 2 we know that a monetary equilibrium with an endogenous debt limit exists at zero interest rates if:

$$[u(\bar{\ell}) - \bar{\ell})] - [u(\tilde{q}) - \tilde{q})] > \frac{\rho \tilde{q}}{\alpha_b}$$

that is, if the difference in the DM surplus between an economy with and without default is big enough. The likelihood that such an equilibrium exists depends on the probabilities of being a patient agent ρ and of being a buyer α_b and DM preferences. See Example 1 and

Figure 5.

Example 1: endogenous debt limit In order to derive intuition for the result in Proposition 2, we consider an example with the functional form $u(q) = \frac{(q+b)^{1-\delta} - b^{1-\delta}}{1-\delta}$ and parameter values $\beta_H = 0.99$, $\beta_L = 0.4$, $\alpha_b = 0.95$, $\rho = 0.1$, $\delta = 0.99999$ and $b = 0.00001$. According to Proposition 2, such an equilibrium exists if $[u(\bar{\ell}) - \bar{\ell}] - [u(\tilde{q}) - \tilde{q}] > \rho\tilde{q}/\alpha_b$, that is, if $[u(\bar{\ell}) - \bar{\ell}] > u(\tilde{q}) - \tilde{q}(\alpha_b - \rho)/\alpha_b$. Figure 5 shows that the equilibrium exists for intermediate values of the debt limit. If the debt limit is sufficiently low, then agents are better off defaulting on the debt, since the debt limit is neutral in that case anyway, and agents can afford $\tilde{q}$ regardless of how it is financed. If the debt limit is sufficiently high, then L agents are better off borrowing once and then defaulting on the debt.

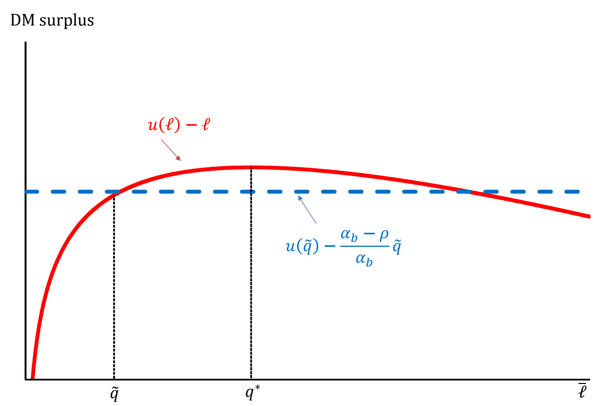

Figure 5: Monetary equilibrium with banking and endogenous debt limit. The horizontal axis reports the endogenous debt limit $\bar{\ell}$. The vertical axis reports the DM surplus. The graph shows the debt limit values for which a monetary equilibrium with banking exists with an endogenous debt limit. According to Proposition 2, such an equilibrium exists if $[u(\ell) - \ell] > u(\tilde{q}) - \tilde{q}(\alpha_b - \rho)/\alpha_b$. The figure is drawn with the utility function $u(q) = ((q+b)^{1-\delta} - b^{1-\delta})/(1-\delta)$, discount factors $\beta_H = 0.99$ and $\beta_L = 0.4$, a measure of buyers $\alpha_b = 0.95$, and a measure of type H agents $\rho = 0.1$, $\delta = 0.99999$ and $b = 0.00001$.

The following result also follows from the proof of Proposition 2.

Corollary 2. *Let $i_a = 0$. With an endogenous borrowing limit, no monetary equilibrium with $m_H > 0$, $m_L > 0$ and banking exists.*

That is, in a banking equilibrium with an endogenous debt limit, impatient agents never bring money at zero interest rates. Why? We know from Proposition 1 that agents of type L bring money only for $\bar{\ell} \leq \tilde{q}$. In that case, however, the debt limit is neutral and agents are indifferent between money and credit. Thus, consumption of impatient agents is the same with and without debt and that is why they have an incentive to default.

8 Conclusion

We investigate the essentiality of money and credit at zero interest rates in a microfounded monetary model in which agents face heterogeneous idiosyncratic time-preference shocks. In this environment, the constrained-efficient allocation is unattainable only with money because equilibrium deflation is bounded by the lowest discount rate, since giving cash a return exceeding the lowest shadow interest rate generates arbitrage opportunities. Thus, impatient agents are always constrained. Three main results arise from our analysis. First, we find that financial intermediation can improve the allocation at zero interest rates because it relaxes the liquidity constraints of impatient borrowers. Second, changes in credit conditions are not necessarily neutral at zero interest rates in a monetary equilibrium. When the debt limit is low, money and credit are perfect substitutes and tightening the debt limit is neutral. As the debt limit increases, however, patient agents keep holding money, while impatient ones prefer not to. Why? Money is costly for them, since they face a positive shadow interest rate, whereas borrowing is costless at zero interest rates. In that case, increasing the debt limit improves the allocation. Third, the welfare-increasing role of banks differs at positive versus zero interest rates. When interest rates are positive, banks provide liquidity insurance. When interest rates are zero, they relax liquidity constraints owing to their ability to enforce debt repayment.

References

Becker, Robert A. 1980. "On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households." *Quarterly Journal of Economics* 95 (2): 375–382. https://doi.org/10.2307/1885506.

Bencivenga, Valerie R., and Gabriele Camera. 2011. "Banking in a Matching Model of Money and Capital." *Journal of Money, Credit and Banking* 43:449–476. https://doi.org/10.1111/j.1538-4616.2011.00446.x.

Berentsen, Aleks, Gabriele Camera, and Christopher J. Waller. 2007. "Money, Credit and Banking." *Journal of Economic Theory* 135 (1): 171–195. https://doi.org/10.1016/j.jet.2006.03.016.

Boel, Paola, and Gabriele Camera. 2006. "Efficient Monetary Allocations and the Illiquidity of Bonds." *Journal of Monetary Economics* 53 (7): 1693–1715. https://doi.org/10.1016/j.jmoneco.2006.02.002.

———. 2020. "Monetary Equilibrium and the Cost of Banking Activity." *Journal of Money, Credit and Banking* 52 (4): 653–683. https://doi.org/10.1111/jmcb.12676.

Boel, Paola, and Christopher J. Waller. 2019. "Stabilization Policy at the Zero Lower Bound." *International Economic Review* 60 (4): 1539–1563. https://doi.org/10.1111/iere.12396.

Chiu, Jonathan, and Césaire A. Meh. 2011. "Financial Intermediation, Liquidity and Inflation." *Macroeconomic Dynamics* 15 (1): 83–118. https://doi.org/10.1017/S1365100510000568.

Diamond, Douglas W., and Philip H. Dybvig. 1983. "Bank Runs, Deposit Insurance, and Liquidity." *Journal of Political Economy* 91 (3): 401–419. https://doi.org/10.1086/261155.

Domínguez, Begoña, and Pedro Gomis-Porqueras. 2019. "The Effects of Secondary Markets for Government Bonds on Inflation Dynamics." *Review of Economic Dynamics* 32:249–273. https://doi.org/10.1016/j.red.2018.10.004.

Dong, Mei, and Stella Huangfu. 2021. "Money and Costly Credit." *Journal of Money, Credit and Banking* 53 (6): 1449–1478. https://doi.org/10.1111/jmcb.12845.

Drugeon, Jean-Pierre, and Bertrand Wigniolle. 2016. "On Time-Consistent Policy Rules for Heterogeneous Discounting Programs." *Journal of Mathematical Economics* 63:174–187. https://doi.org/10.1016/j.jmateco.2016.01.006.

Falk, Armin, Anke Becker, Thomas Dohmen, Benjamin Enke, David Huffman, and Uwe Sunde. 2018. "Global Evidence on Economic Preferences." *The Quarterly Journal of Economics* 133 (4): 1645–1692. https://doi.org/10.1093/qje/qjy013.

Ferraris, Leo, and Fabrizio Mattesini. 2020. "Money and Collateral." *Journal of Money, Credit and Banking* 52 (7): 1617–1644. https://doi.org/10.1111/jmcb.12724.

Ferraris, Leo, and Makoto Watanabe. 2008. "Collateral Secured Loans in a Monetary Economy." *Journal of Economic Theory* 143 (1): 405–424. https://doi.org/10.1016/j.jet.2008.02.002.

———. 2011. "Collateral Fluctuations in a Monetary Economy." *Journal of Economic Theory* 146 (5): 1915–1940. https://doi.org/10.1016/j.jet.2011.06.012.

Gomis-Porqueras, Pedro, Adrian Peralta-Alva, and Christopher Waller. 2014. "The Shadow Economy as an Equilibrium Outcome." *Journal of Economic Dynamics and Control* 41:1–19. https://doi.org/10.1016/j.jedc.2014.02.006.

Gomis-Porqueras, Pedro, and Daniel Sanches. 2013. "Optimal Monetary Policy in a Model of Money and Credit." *Journal of Money, Credit and Banking* 45 (4): 701–729. https://doi.org/10.1111/jmcb.12021.

Gu, Chao, Fabrizio Mattesini, and Randall Wright. 2016. "Money and Credit Redux." *Econometrica* 84:1–32. https://doi.org/10.3982/ECTA12798.

Kocherlakota, Narayana R. 1998. "Money is Memory." *Journal of Economic Theory* 81 (2): 232–251. https://doi.org/10.1006/jeth.1997.2357.

Lagos, Ricardo, and Randall D. Wright. 2005. "A Unified Framework for Monetary Theory and Policy Analysis." *Journal of Political Economy* 113 (3): 463–484. https://doi.org/10.1086/429804.

Lawrance, Emily C. 1991. "Poverty and the Rate of Time Preference: Evidence from Panel Data." *Journal of Political Economy* 99 (1): 54–77. https://doi.org/10.1086/261740.

Lotz, Sébastien, and Cathy Zhang. 2016. "Money and Credit as Means of Payment: A New Monetarist Approach." *Journal of Economic Theory* 164:68–100. https://doi.org/10.1016/j.jet.2015.08.003.

Samwick, Andrew A. 1998. "Discount Rate Heterogeneity and Social Security Reform." *Journal of Development Economics* 57 (1): 117–146. https://doi.org/10.1016/S0304-3878(98)00080-7.

Telyukova, Irina A., and Randall Wright. 2008. "A Model of Money and Credit, with Application to the Credit Card Debt Puzzle." *The Review of Economic Studies* 75 (2): 629–647. https://doi.org/10.1111/j.1467-937X.2008.00487.x.

Weidner, Justin, Greg Kaplan, and Giovanni Violante. 2014. "The Wealthy Hand-to-Mouth." *Brookings Papers on Economic Activity* 48:77–153. https://www.brookings.edu/bpea-articles/the-wealthy-hand-to-mouth/.

Appendix 1: Proofs

Proof of Lemma 1 From the Euler equation in (31) we have that the following must hold:

$$\beta_L + \lambda_L^a = \beta_H + \lambda_H^a$$

Since $\beta_H > \beta_L$, it must be that $\lambda_L^a > \lambda_H^a \geq 0$. If $\lambda_L^a > \lambda_H^a > 0$, then there is no borrowing or lending. If instead $\lambda_L^a > \lambda_H^a = 0$, then $a_L = A$ and given the bond market clearing condition (23) we have that $a_H = -A(1-\rho)/\rho$. Since $\pi p_a = \beta_H$ from (31), then $p_a = \beta_H/\pi$. $\square$

Proof of Lemma 2 Consider the first-order conditions in the DM for an agent who experienced a shock $k = s$ in the DM. Since $\lambda_j^s \geq 0$, then from (18) it must be that $i_d \geq 0$. Since the banking sector is competitive, $i = i_d$ and therefore $i \geq 0$. Suppose $\pi < \beta_H$. Then $(\pi - \beta_H)/\beta_H < 0$, which from (30) is incompatible with a monetary equilibrium for type H agents if $i \geq 0$. Thus, it must be that $\pi \geq \beta_H$. $\square$

Proof of Lemma 3 We now want to provide conditions that guarantee $n_{jz}^k \geq 0$ when $i_a = 0$. Note that if $n_{HL}^s \geq 0$, then $n_{jz}^k \geq 0$ in all other cases. We know that $x_{jz}^k = x^*$ for all j, z. This, together with the budget constraint in (8), implies:

$$n_{HL}^s = x^* - m_H^s + \pi m_L - \pi p_a a_L + a_H - \tau - (1+i)d_H$$

From (5), Lemma 1, and (7) the expression above becomes:

$$n_{HL}^s = x^* - m_H - py + d_H + \beta_H m_L - A[\beta_H + (1-\rho)/\rho] - \tau - (1+i)d_H$$

We know that in equilibrium $p = 1$ and $y = \rho m_H + (1-\rho)m_L$. Moreover, when $i = 0$ we have that $\tau = (\beta_H - 1)(\rho m_H + (1-\rho)m_L)$ and $m_H = q^*$. So, rearranging we get:

$$n_{HL}^s = x^* - A[\beta_H + (1-\rho)/\rho] - q^*(1+\rho\beta_H) + \rho\beta_H m_L > x^* - A[\beta_H + (1-\rho)/\rho] - (1+\rho\beta_H)q^*$$

Therefore, $n_{HL}^s \geq 0$ if the following condition is satisfied:

$$A \leq \frac{x^* - (1+\rho\beta_H)q^*}{\beta_H + (1-\rho)/\rho} \tag{43}$$

which implies that if the borrowing constraint is tight enough, then an incentive to work is generated in the CM. $\square$

Proof of Proposition 1 Let $\pi = \beta_H$ and therefore $i_a = 0$. Consider a type H. In order for condition (34) to be satisfied for $j = H$, it must be that $i = 0$, $\mu_H^m = 0$ and $\lambda_H^\ell = 0$ and therefore $m_H > 0$ and $q_H = q^*$ from (20) and (21).

Consider now a type L. We know from (34) that if $\pi = \beta_H$ then $i = 0$. If $\pi = \beta_H$, then $(\pi - \beta_L)/\beta_L > 0$. Therefore, from (30) we know that in order for (33) to hold for $j = L$ at $\pi = \beta_H$ one of the following three cases has to be true: (i) $\lambda_L^\ell > 0$ and $\mu_L^m = 0$; (ii) $\lambda_L^\ell > 0$ and $\mu_L^m > 0$; or (iii) $\lambda_L^\ell = 0$ and $\mu_L^m > 0$. In case (i), $m_L > 0$ and $q_L = \tilde{q} < q^*$ where $\tilde{q}$ solves (33) for $j = L$ with $\mu_L^m = 0$. In case (ii), $m_L = 0$ and $q_L < q^*$. In case (iii), $m_L = 0$ and $q_L = q^*$. For cases (ii) and (iii), if $m_L = 0$ then we know from the DM buyer's problem that $\alpha_b(1-\rho)q_L^b = \rho(1-\alpha_b)m_H$, since the only agents borrowing are type L agents, and type H sellers and idle agents are the only depositors with $m_H > 0$. Hence, $q_L < q^*$ if $\rho < \alpha_b$, and $q_L = q^*$ otherwise. $\square$

Proof of Corollary 1 The proof follows directly from Proposition 1 and the fact that deposit rates are zero when $i_a = 0$. $\square$

Proof of Proposition 2 We first focus on the case $j = L$ and $z = H$, that is the case of an agent who is a borrower in this period and won't need to borrow in the next one. We will then examine the case $j = H$ and $z = H$. The real borrowing constraint $\bar{\ell}$ must satisfy (40), and combining this with (41) and (42) we have:

$$U(x^*) - n_{LH}^b + \beta_H V(\omega_H') = U(\hat{x}_{LH}^b) - \hat{n}_{LH}^b + \beta_H \hat{V}(\hat{\omega}_H') \tag{44}$$

If the buyer repays her loans then she will have to work:

$$\begin{aligned} n_{LH}^b &= x^* + \pi m_H' - (m_L^b + \tau) + (1+i)\bar{\ell} + a_L - p_a \pi a_H' \\ &= x^* + \pi m_H' - (m_L - q_L + \bar{\ell} + \tau) + (1+i)\bar{\ell} + a_L - \beta_H a_H' \\ &= x^* + \pi m_H' - (m_L + \tau) + q_L + i\bar{\ell} + a_L - \beta_H a_H' \end{aligned}$$

Since $\tau = (\pi - 1)\bar{m}$ where $\bar{m} = \rho m_H + (1-\rho)m_L$, then the expression above becomes:

$$n_{LH}^b = x^* + \pi m_H' - m_L - (\pi - 1)\bar{m} + q_L + i\bar{\ell} + a_L - \beta_H a_H' \tag{45}$$

If an agent instead decides to default on her loans, then she will have to work:

$$\begin{aligned} \hat{n}_{LH}^b &= \hat{x}_{LH}^b + \pi \hat{m}_H' - (m_L^b + \tau) + a_L - p_a \pi \hat{a}_H' \\ &= \hat{x}_{LH}^b + \pi \hat{m}_H' - (m_L - q_L + \bar{\ell} + \tau) + a_L - p_a \pi \hat{a}_H' \\ &= \hat{x}_{LH}^b + \pi \hat{m}_H' - (m_L + \tau) + q_L - \bar{\ell} + a_L - p_a \pi \hat{a}_H' \end{aligned}$$

and therefore

$$\hat{n}_{LH}^b = \hat{x}_{LH}^b + \pi \hat{m}_H' - m_L - (\pi - 1)\bar{m} + q_L - \bar{\ell} + a_L - p_a \pi \hat{a}_H'$$

The defaulting problem in the CM for a buyer in the DM, therefore, is:

$$\hat{W}_H(\omega_L^b) = \underset{\hat{x}_{LH}^b,\hat{m}_H',\hat{n}_{LH}^b}{Max} U(\hat{x}_{LH}^b) - \hat{n}_{LH}^b + \beta_H \hat{V}(\hat{\omega}_H')$$
$$\text{s.t. } \hat{n}_{LH}^b = \hat{x}_{LH}^b + \pi\hat{m}_H' - m_L - (\pi-1)\bar{m} + q_L - \bar{\ell} + a_L - \beta_H \hat{a}_H'$$

The first-order conditions for the problem above are $\beta_H \partial\hat{V}/\partial\hat{m}_H' + \hat{\mu}_z^m = \pi$, $\beta_H \partial\hat{V}/\partial\hat{a}_H' - \hat{\lambda}_z^a = -\pi p_a$ and $U'(\hat{x}_{LH}^b) = 1$ so that $\hat{x}_{LH}^b = x^*$. This implies:

$$\hat{n}_{LH}^b = x^* + \pi\hat{m}_H' - m_L - (\pi-1)\bar{m} + q_L - \bar{\ell} + a_L - \beta_H \hat{a}_H' \tag{46}$$

Since $a_H' = \hat{a}_H'$ from (9), combining (44), (45), and (46) we have that the following condition must hold:

$$(1+i)\bar{\ell} = \pi(\hat{m}_H' - m_H') + \beta_H[V(\omega_H') - \hat{V}(\hat{\omega}_H')] \tag{47}$$

Since in a stationary equilibrium $\hat{m}_H' = \hat{m}_H$, $m_H' = m_H$, $\hat{\omega}_H' = \hat{\omega}_H$ and $\omega_H' = \omega_H$, the condition above becomes:

$$(1+i)\bar{\ell} = \pi(\hat{m}_H - m_H) + \beta_H[V(\omega_H) - \hat{V}(\hat{\omega}_H)] \tag{48}$$

Note that if the deviator is a seller in the DM, then her production choice has to satisfy $c'(\hat{y}) = 1$. Thus, off the equilibrium path, both deviating and non-deviating sellers will produce the same amount $\hat{y} = y$. Therefore, the continuation payoffs are:

$$V(\omega_H) = m_H - a_H + \alpha_b[u(q_H) - q_H - i\ell_H] + i(1-\alpha_b)d_H + EV + U(x^*) - x^* \tag{49}$$

and

$$\hat{V}(\hat{\omega}_H) = \hat{m}_H - \hat{a}_H + \alpha_b[u(\hat{q}_H) - \hat{q}_H] + E\hat{V} + U(x^*) - x^* \tag{50}$$

where $EV = \rho\beta_H V(\omega_H) + (1-\rho)\beta_L V(\omega_L)$ and $E\hat{V} = \rho\beta_H \hat{V}(\hat{\omega}_H) + (1-\rho)\beta_L \hat{V}(\hat{\omega}_L)$. Combining (47), (49), and (50) we have that the real borrowing constraint $\bar{\ell}$ has to satisfy:

$$(1+i)\bar{\ell} = (\pi - \beta_H)(\hat{m}_H - m_H) + \beta_H\{\Psi(q_H, \hat{q}_H) + [(1-\alpha_b)d_H - \alpha_b\bar{\ell}]i + EV - E\hat{V}\} \tag{51}$$

where $\Psi(q_H, \hat{q}_H) = \alpha_b[u(q_H) - u(\hat{q}_H) - (q_H - \hat{q}_H)]$. From (50) we know that the marginal value of money for a deviator is:

$$\frac{\partial\hat{V}(\hat{\omega}_H)}{\partial\hat{m}_H} = 1 + \alpha_b\left[u'(\hat{q}_H) - 1\right] \tag{52}$$

and therefore:

$$\frac{\pi - \beta_H}{\beta_H} = \alpha_b \left[u'(\hat{q}_H) - 1\right] + \frac{\hat{\mu}_H^m}{\beta_H} \tag{53}$$

If $\pi = \beta_H$ then (51) can be expressed as:

$$\bar{\ell} = (\pi - \beta_H)(\hat{m}_H - m_H) + \beta_H \left[\Psi(q_H, \hat{q}_H) + EV - E\hat{V}\right] \tag{54}$$

The right-hand side of (54) must be positive in order for an equilibrium with credit to exist. By comparing (30) and (53), we know that if $\pi = \beta_H$ and therefore $i = 0$, then $\hat{\mu}_H^m = 0$ and $\hat{q}_H = q_H = q^*$. Thus, we have that $\bar{\ell} = \beta_H[EV - E\hat{V}] = \rho\beta_H[V(\omega_H) - \hat{V}(\hat{\omega}_H)] + (1-\rho)\beta_L[V(\omega_L) - \hat{V}(\hat{\omega}_L)]$. Since $V(\omega_H) = \hat{V}(\hat{\omega}_H)$ given that we are considering a one-shot deviation for type L agents, then in order to have $\bar{\ell} > 0$ we need $V(\omega_L) > \hat{V}(\hat{\omega}_L)$. We now want to prove when that's the case. We have:

$$V(\omega_L) = \frac{U(x^*) - x^* - \rho(m_H - m_L) - a_L + \alpha_b[u(q_L) - q_L] + \rho\beta_H V(\omega_H)}{1 - \beta_L(1-\rho)}$$

$$\hat{V}(\hat{\omega}_L) = \frac{U(x^*) - x^* - \rho(m_H - \hat{m}_L) - \hat{a}_L + \alpha_b[u(\hat{q}_L) - \hat{q}_L] + \rho\beta_H V(\omega_H)}{1 - \beta_L(1-\rho)}$$

Now we use $\hat{a}_L = a_L$. That is because $\partial\hat{V}/\partial\hat{a}_j = -1$ and therefore the proof is the same as in Lemma 1 in the paper. So, we have $\hat{a}_L = a_L = A$, $\hat{a}_H = a_H = -(1-\rho)A/\rho$ and $\hat{p}_a = p_a = \beta_H/\pi$. Therefore, in order for $V_L(\omega_L) > \hat{V}(\hat{\omega}_L)$ to hold, we need:

$$\rho(m_L - \hat{m}_L) + \alpha_b[u(q_L) - q_L] - \alpha_b[u(\hat{q}_L) - \hat{q}_L] > 0$$

We know that $q_L = m_L + \bar{\ell}$ and $\hat{q}_L = \hat{m}_L = \tilde{q}$ where $\tilde{q}$ is consumption in the equilibrium in Proposition 1 in which L agents bring money. In that case, we know from Proposition 1 that agents are indifferent between using money or borrowing. If that is the case, then agents who don't default (and can use money or loans) and agents who do default (and can only bring money) will consume the same amount, which is $\tilde{q}$. Therefore, the inequality above can be rewritten as $\rho(q_L - \bar{\ell} - \tilde{q}) + \alpha_b[u(q_L) - q_L] - \alpha_b[u(\tilde{q}) - \tilde{q}] > 0$ so that:

$$\bar{\ell} < \frac{\alpha_b[u(q_L) - u(\tilde{q})] + (\rho - \alpha_b)(q_L - \tilde{q})}{\rho} \tag{55}$$

Notice that in the case in which $m_L > 0$, we know from Proposition 1 that $q_L = \hat{q}_L = \tilde{q}$. Then, (55) implies $\bar{\ell} < 0$.

Now consider the case $m_H > m_L = 0$, in which case $q_L > \tilde{q}$ from Proposition 1. Consider the right-hand side in (55) and note that $(\alpha_b[u(q_L) - u(\tilde{q})] + (\rho - \alpha_b)(q_L - \tilde{q}))/\rho >$

$\alpha_b[u(q_L) - u(\tilde{q})] + (\rho - \alpha_b)(q_L - \tilde{q}) > \alpha_b[u(q_L) - u(\tilde{q})] - \alpha_b(q_L - \tilde{q})$. So, $\bar{\ell} > 0$ if $\alpha_b[u(q_L) - u(\tilde{q})] - \alpha_b(q_L - \tilde{q}) > 0$:

$$\begin{aligned}
&\alpha_b[u(q_L) - u(\tilde{q})] - \alpha_b(q_L - \tilde{q}) > 0 \\
\Rightarrow &[u(q_L) - u(\tilde{q})] - (q_L - \tilde{q}) > 0 \\
\Rightarrow &\frac{u(q_L) - u(\tilde{q})}{q_L - \tilde{q}} - 1 > u'(q_L) - 1 \geq 0 \text{ from (20)}
\end{aligned}$$

and therefore $\bar{\ell} > 0$.

Now we focus on the case $j = H$ and $z = H$. The real borrowing constraint $\bar{\ell}$ must satisfy (40) also in this case, and combining this with (41) and (42) we have:

$$U(x^*) - n^b_{HH} + \beta_H V(\omega'_H) = U(\hat{x}^b_{HH}) - \hat{n}^b_{HH} + \beta_H \hat{V}(\hat{\omega}'_H) \tag{56}$$

If the buyer repays her loans then she will have to work:

$$\begin{aligned}
n^b_{HH} &= x^* + \pi m'_H - (m^b_H + \tau) + (1+i)\bar{\ell} + a_H - p_a \pi a'_H \\
&= x^* + \pi m'_H - (m_H - q_* + \bar{\ell} + \tau) + (1+i)\bar{\ell} + a_H - \beta_H a'_H \\
&= x^* + \pi m'_H - (m_H + \tau) + q_H + i\bar{\ell} + a_H - \beta_H a'_H
\end{aligned}$$

Since $\tau = (\pi - 1)\bar{m}$ where $\bar{m} = \rho m_H + (1-\rho) m_L$, then the expression above becomes:

$$n^b_{HH} = x^* + \pi m'_H - m_H - (\pi - 1)\bar{m} + q^* + i\bar{\ell} + a_H - \beta_H a'_H \tag{57}$$

If an agent instead decides to default on her loans, then she will have to work:

$$\begin{aligned}
\hat{n}^b_{HH} &= \hat{x}^b_{HH} + \pi \hat{m}'_H - (m^b_H + \tau) + a_H - p_a \pi \hat{a}'_H \\
&= \hat{x}^b_{HH} + \pi \hat{m}'_H - (m_H - q^* + \bar{\ell} + \tau) + a_H - p_a \pi \hat{a}'_H \\
&= \hat{x}^b_{HH} + \pi \hat{m}'_H - (m_H + \tau) + q^* - \bar{\ell} + a_H - p_a \pi \hat{a}'_H
\end{aligned}$$

and therefore

$$\hat{n}^b_{HH} = \hat{x}^b_{HH} + \pi \hat{m}'_H - m_H - (\pi - 1)\bar{m} + q^* - \bar{\ell} + a_H - p_a \pi \hat{a}'_H$$

The defaulting problem in the CM for a buyer in the DM, therefore, is:

$$\begin{aligned}
&\hat{W}_H(\omega^b_L) = \underset{\hat{x}^b_{HH}, \hat{m}'_H, \hat{n}^b_{HH}}{Max} \; U(\hat{x}^b_{HH}) - \hat{n}^b_{HH} + \beta_H \hat{V}(\hat{\omega}'_H) \\
&\text{s.t. } \hat{n}^b_{HH} = \hat{x}^b_{HH} + \pi \hat{m}'_H - m_H - (\pi - 1)\bar{m} + q^* - \bar{\ell} + a_H - \beta_H \hat{a}'_H
\end{aligned}$$

The first-order conditions for the problem above are $\beta_H \partial \hat{V}/\partial \hat{m}'_H + \hat{\mu}^m_z = \pi$, $\beta_H \partial \hat{V}/\partial \hat{a}'_H - \hat{\lambda}^a_z =$

$-\pi p_a$ and $U'(\hat{x}^b_{HH}) = 1$ so that $\hat{x}^b_{HH} = x^*$. This implies:

$$\hat{n}^b_{HH} = x^* + \pi\hat{m}'_H - m_H - (\pi - 1)\bar{m} + q^* - \bar{\ell} + a_H - \beta_H\hat{a}'_H \tag{58}$$

Since $a'_H = \hat{a}'_H$, combining (56), (57), and (58) we have that the following condition must hold:

$$(1 + i)\bar{\ell} = \pi(\hat{m}'_H - m'_H) + \beta_H[V(\omega'_H) - \hat{V}(\hat{\omega}'_H)] \tag{59}$$

Since in a stationary equilibrium $\hat{m}'_H = \hat{m}_H$, $m'_H = m_H$, $\hat{\omega}'_H = \hat{\omega}_H$ and $\omega'_H = \omega_H$, the condition above becomes:

$$(1 + i)\bar{\ell} = \pi(\hat{m}_H - m_H) + \beta_H[V(\omega_H) - \hat{V}(\hat{\omega}_H)] \tag{60}$$

Since (60) coincides with (48), then $\bar{\ell}$ will be defined by (55) also in this case. $\square$

Proof of Corollary 2. The proof follows directly from the one of Proposition 2. $\square$