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On the Economic Mechanics of Warfare*

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Abstract

A large literature is concerned with the consequences of war-related expenditures and how to finance them. Yet, there is little by way of understanding how expenditures affect the outcomes of wars, e.g., prevailing side, duration, or total destruction. I present a model of attrition in which I characterize the effects of resources on the outcomes of war for a military conclusion (when one side cannot fight anymore) and a political conclusion (when one side does not want to fight anymore). I discuss the role of GDP for both types of conclusion. I also analyze the mechanics of third-party support to a small country at war with a large one, e.g., Ukraine and Russia. Finally, I show that the model can fit actual battle data.

JEL: E6, H56, N4
Keywords: War; Attrition; Military spending.

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1 Introduction

War may be the costliest activity countries regularly engage in. Naturally, then, there is a sizeable literature concerned with the optimal financing of war expenditures.[1] There is little, however, by way of understanding how expenditures affect the outcomes of war such as duration, destruction (material and human), the prevailing side, and the reason for the conclusion of war. This is what this article is about: A theory of how resources affect the outcomes of war.

The outcomes of war are of interest to economists. Duration determines the financial cost and has economic, demographic, and political consequences besides the cost. When, for example, a war reduces fertility (e.g., Caldwell, 2004, Vandenbroucke, 2014), disrupts the marriage market (e.g., Knowles and Vandenbroucke, 2019, Abramitzky et al., 2011), and disrupts the labor market (e.g., Acemoglu et al., 2004, Fernández, 2013, Doepke et al., 2015), the effects are magnified by duration.

The destruction, either material or human (casualties) and, associated to it, the resources remaining at the end of a war, are outcomes of interest as well. First, the material destruction has consequences for well-being (e.g., as with the destruction of housing), productivity (e.g., as with the destruction of productive capital), and sometimes international relations (e.g., the Marshall Plan). Casualties also matter for well-being and the age and sex composition of the labor force in the postwar years. Second, at the end of a war, a country's remaining resources, such as its military equipment, can be sold, either as scrap or to another country purchasing armament on the secondary market. Individuals (troops) can work and be productive after they are demobilized. Remaining resources may also have a military value if they can serve for peacekeeping and/or deterrence purposes.

The determination of the prevailing side is of interest chiefly because the prevailing side often imposes a payment on its opponent in the form of land (e.g., any war of conquest), treasure (e.g., the treaty of Versailles), and sometimes people (i.e., slaves). Also of interest is the reason why one side ceases to fight and may accept such payment. It can be for a lack of resources to fight (e.g., Germany and Japan at the

[1] See, for instance, Keynes (1940), Barro (1979), Lucas and Stokey (1983), and more recently, Ohanian (1997) and McGrattan and Ohanian (2010)

end of World War II) or for a lack of political will (e.g., the U.S. in Vietnam).

To analyze the role of resources in a war, I present a model of how a war unfolds and concludes. That is what I mean by "warfare." It is not a "model of war" explaining why countries go to war, however interesting this question is. Instead, I assume a state of war to exist between two countries. Countries are endowed with initial "weapons" stocks and military technologies using weapons to destroy their opponent's weapons. They also receive per-period exogenous flows of weapons ("reinforcements").

I interpret initial weapons stock as resulting from prewar investments, and reinforcements as resulting from wartime expenditures. These, together with the military technologies, are the resources the effects of which I seek to analyze. I interpret weapons as combinations of physical and human capital. There are no decisions.

The destruction of resources experienced by countries over the course of a war is mitigated by their reinforcements and the destruction they inflict on their opponents. The model's outcome is thus the joint dynamics of weapons accumulation, i.e., the countries' ability to fight, and of weapons destruction and casualties. I discuss two possible conclusions: In the first, which I label "military," a war concludes when a country's weapons stock reaches an exogenously-determined low threshold and the country cannot fight anymore. In the second, which I label "political," a war concludes when a country's casualties reach an exogenously-determined high threshold and political forces request that the fighting ends.

Observations about the theory

Battles v. Attrition Wars are sometime viewed as idiosyncratic events with outcomes largely determined by genius-like generalship and "decisive" battles. I do not take this view. My approach is inspired, instead, by observations from military historians on the one hand, and by Operations Research models on the other hand.

Historians such as O'Brien (2015) and Nolan (2017) argue against a battle-centric view of wars. O'Brien's first sentence is "*There were no decisive battles in World War II*" (p. 1). Nolan insists that attrition, more than battles, is key to understanding the outcomes of conflict such as the Punic wars, the Hundred Years war, the Napoleonic wars, and the two World Wars of the Twentieth century. In the same vein, Parshall and Tully (2005), in their comprehensive study of the battle of Midway, argue that, although important, Midway was not decisive.[2]

In the last analysis, the notion of a decisive battle is not a useful one. That is first because "decisive" is not well-defined. What is the decided outcome? Is it the prevailing country, the duration of the war, or another outcome? Second, regardless of the definition, assessing whether a battle is decisive requires a counterfactual, which cannot be evaluated without a theory of decision making in war. The latter does not exist to the best of my knowledge. (I discuss decision making below.) For these reasons, the model I present in this paper does not have battles in the sense of discrete occurrences of fighting, some possibly more significant than others. Instead, I represent the war as a process of attrition taking place at each point in time.

Combat models developed in Operations Research since the advent of the so-called Lanchester model (Lanchester, 1916) are systems of differential equations describing the attrition of opposing forces during battles. For the reasons I described above, such emphasis on attrition in Lanchester-type models makes these models appealing to think about war as a whole. That is therefore my approach: I use a Lanchester-type model of combat and interpret it as a representation of war.

Decisions I do not model decisions. The model represents the mechanics of attrition, as the Solow model represents the mechanics of capital accumulation. Absent a theory of why there is a war, it is difficult to assign objective functions to the belligerents. Although interesting, I abstract from these considerations here.

Without decisions, there are no tactics and/or strategies. I consider tactical knowledge as one would consider business knowledge in a model featuring production functions: I assume tactical knowledge to be subsumed in the military technology.

[2] They write "*(...) win or loose at Midway, the vast industrial resources of the United States gave its navy an absolutely irrevocable writ of strategic dominance in the Pacific War.*" (p. 424) and "*(...) Midway stands as the most important battle of the Pacific War, not because it was decisive in an absolute sense, and not because it won the war in a day, but because of its (...) effects on American military options in the Pacific.*" (p. 430).

The organization of the paper

I develop the model in Section 2. I layout the setup in 2.1 and describe the dynamics of weapons accumulation and casualties in 2.2. In 2.3 I describe the model's characterization of a military conclusion. Specifically, I describe how resources affect which side prevails, the duration, the remaining resources, and total casualties at the end of a war. In 2.4 I characterize a political conclusion.

In Section 3 I show how three historical scenarios can be viewed through the lenses of the model. The first is inspired by Japan declaring war on the United States in December 1941 and by Russia invading Ukraine in February 2022. Both cases feature two countries with large differences in their Gross Domestic Product. In 3.1 I reinterpret the model in terms of Gross Domestic Product, prewar, and wartime saving rates to discuss this scenario. The second scenario is inspired by the Phoney war in Western Europe in 1940 and by the 11-month delay between the German declaration of war on the United States in December 1941 and the first major operation by U.S. forces against German forces in November 1942. In 3.2 I discuss the implications of waiting before the start of hostilities. The last scenario is inspired, again, by the 2022 invasion of Ukraine by Russia. In 3.3 I analyze the mechanics of third-party (e.g., an international coalition) support to a small country at war with a large one.

In Section 4 I replicate an application of the model to the battle of Iwo Jima during World War II. I discuss the data needed for an estimation and show that in the case of the battle of Iwo Jima, the model fits the data well.

Some results

In Section 2.3 I find that, at a military conclusion, casualties on **both** sides are decreasing with the resources committed by the country obtaining the military victory, and increasing with the resources committed by its opponent. That is because the country obtaining the military victory can shorten the war by allocating more resources to it, thereby reducing destruction and casualties for **both** sides.

In Section 3.1, I describe the role of the relative GDP of belligerents. All else equal, a high GDP makes the condition for a military victory more favorable **and** the condition for suing for peace on political grounds less favorable. The first effect is often mentioned by military historians (e.g., footnote 2). The second effect is less often described: The richer a country, the longer it takes to reach the political threshold for casualties (at which political forces request an end to the fighting) and thus the more time there is for obtaining a favorable military conclusion.

In Section 3.3, I describe the mechanics of third-party support to a small country at war with a large one, e.g., Ukraine v. Russia. When the foreign coalition commits an additional unit of per-period reinforcements, the war becomes shorter. Hence, there is a well-defined cost-minimizing level of support.

2 The model

2.1 Setup

The model I present is inspired by the so-called Lanchester model (Lanchester, 1916). Time is continuous and there is no uncertainty. There are two countries, Red and Blue, with weapons stocks denoted K_t^R and K_t^B, respectively. A country's weapons stock is an input into its military "production function," the output of which is a flow of destruction inflicted on the opposite side's weapons stock. Let $\theta^R \in (0,1)$ denote Red's attrition coefficient, which is the flow of Blue weapons destroyed per Red weapon at each point in time. Blue's attrition coefficient, $\theta^B \in (0,1)$, has a similar interpretation. Let $X^R \geq 0$ and $X^B \geq 0$ denote constant reinforcement flows at each point in time. I do not model depreciation for simplicity. The laws of motion for weapons stocks are thus

$$
\begin{aligned}
dK_t^R/dt &= -\theta^B K_t^B + X^R, && (1)\\
dK_t^B/dt &= -\theta^R K_t^R + X^B, && (2)
\end{aligned}
$$

where the initial conditions K_0^R and K_0^B are given. In what follows I use the term "resources" to refer to initial weapons stocks and/or reinforcements.

I interpret a unit of weapon as a combination of human and material resources, e.g., a soldier with a rifle or an aircraft with a crew, etc. I assume a high degree of complementarity between human and material resources at the weapon level. Under this assumption, I interpret the destruction of a unit of weapon as a combination of casualties and material destruction. (I will use these terms interchangeably). I do not distinguish between lethal and non-lethal casualties. It is possible that one aircraft with crew effects as much destruction as a large number of soldiers with rifles, suggesting a degree of substitutability between different weapon types. This is not a contradiction to the notion of complementarity between human and material resources for a weapon type. In the model there is one aggregate weapon type for simplicity. I argue that this is innocuous: My goal, to analyze the role of resources in the outcomes of war, is not better served by modeling multiple weapon types.

I assume there is no destruction other than that of weapons and, thus, abstract from some form of destruction found in war. First, since resources are endowments, there is no destruction of the productive capacities of a country to reduce its ability to reinforce, e.g., no bombing of factories or blockading of ports. Second, there are no civilians. Casualties result only from the destruction of weapons, e.g., no collateral damages or direct targeting of civilians. I present, in Appendix H, a version of the model with civilian casualties and show that the resulting dynamics is isomorphic to that of the model without civilians.

The steady state $(\bar{K}^R, \bar{K}^B)$ of system (1)-(2) is a stalemate such that a country's reinforcements are exactly destroyed by the other country: $\bar{K}^R = X^B/\theta^R$ and $\bar{K}^B = X^R/\theta^B$. Define $\tilde{K}_t^R = K_t^R - \bar{K}^R$ and $\tilde{K}_t^B = K_t^B - \bar{K}^B$. Then, (1)-(2) become

$$\begin{pmatrix} d\tilde{K}_t^R/dt \\ d\tilde{K}_t^B/dt \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -\theta^B \\ -\theta^R & 0 \end{pmatrix}}_{\mathcal{M}} \begin{pmatrix} \tilde{K}_t^R \\ \tilde{K}_t^B \end{pmatrix}. \tag{3}$$

Let λ_1 and λ_2 be the eigenvalues of $\mathcal{M}$ with corresponding eigenvectors $[1, v_1]'$ and $[1, v_2]'$, respectively:

$$\begin{aligned} \lambda_1 &= -\sqrt{\theta^R \theta^B}, \qquad & v_1 &= \sqrt{\theta^R/\theta^B}, \\ \lambda_2 &= \sqrt{\theta^R \theta^B}, \qquad & v_2 &= -\sqrt{\theta^R/\theta^B}. \end{aligned}$$

2.2 Dynamics

Weapons stocks Standard methods (Appendices A and B) yield the solution

$$\tilde{K}_t^R = \frac{1}{2}\left[e^{t\lambda_1}\mathcal{A} - e^{t\lambda_2}\mathcal{B}\right]\frac{1}{v_1}, \tag{4}$$

$$\tilde{K}_t^B = \frac{1}{2}\left[e^{t\lambda_1}\mathcal{A} + e^{t\lambda_2}\mathcal{B}\right], \tag{5}$$

where the constant $\mathcal{A}$ and $\mathcal{B}$ depend on initial conditions:

$$\mathcal{A} = \tilde{K}_0^B + v_1\tilde{K}_0^R \text{ and } \mathcal{B} = \tilde{K}_0^B - v_1\tilde{K}_0^R.$$

Note that $\lambda_1 < 0$ and $\lambda_2 > 0$. Thus, the stalemate is a saddle-point in the (K_t^R, K_t^B) state space. There is a stable branch described by $\mathcal{B} = 0$ and an unstable branch described by $\mathcal{A} = 0$. Note also, from Equation (3), that $dK_t^B/dt > 0$ whenever $K_t^R < \bar{K}^R$. That is because when the Red weapons stock is below its stalemate, Red does not offset Blue reinforcements and, thus, the Blue weapons stock increases. Conversely, if the Red weapons stock is above its stalemate, Blue reinforcements do not offset the destruction caused by Red and, thus, the Blue weapons stock decreases: $dK_t^B/dt < 0$ whenever $K_t^R > \bar{K}^R$. The same logic applies to the evolution of the Red weapons stock. Figure 1 summarizes these observations in a phase diagram.

Weapons stocks' trajectories need not be monotonic. The blue arrow starting off in the light-shaded area of Figure 1 represents a case where both weapons stocks are initially below their stalemate values: The Blue weapons stock increases monotonically; the Red weapons stock increases until $K_t^B = \bar{K}^B$, when $t = \ln(-\mathcal{B}/\mathcal{A})/(2\lambda_1)$, and then decreases. It is the reverse with the blue arrow starting off in the dark-shaded area: The Red weapons stock decreases monotonically; the Blue weapons stock decreases until $K_t^R = \bar{K}^R$, when $t = \ln(\mathcal{B}/\mathcal{A})/(2\lambda_1)$, and then increases. Finally, when initial conditions are in the northwest quadrant, weapons stocks evolve monotonically: upward for Blue and downward for Red. A similar analysis holds for initial conditions below the stable branch.

Casualties The flow of casualties, at each point in time, are $d_t^B = \theta^R K_t^R$ for Blue and $d_t^R = \theta^B K_t^B$ for Red. I show (Equations B.7 and B.9 in Appendix B) that the

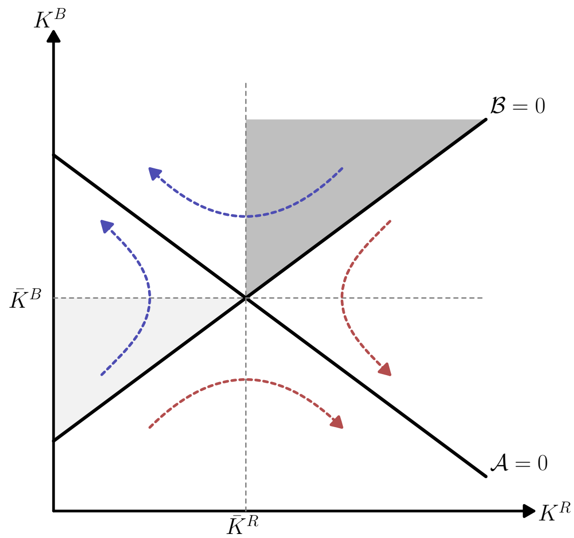


Figure 1: Weapons stock dynamics in the Lanchester model

Red weapons stock is decreasing with the initial Blue weapons stock and with Blue reinforcements. It follows that d_t^B is decreasing in K_0^B and X^B:

$$\frac{\partial K_t^R}{\partial X^B} < 0 \Rightarrow \frac{\partial d_t^B}{\partial X^B} < 0 \quad \text{and} \quad \frac{\partial K_t^R}{\partial K_0^B} < 0 \Rightarrow \frac{\partial d_t^B}{\partial K_0^B} < 0. \tag{6}$$

I also show (Equations B.3 and B.5 in Appendix B) that the Blue weapons stock is increasing with K_0^B and X^B, implying that d_t^R is increasing as well:

$$\frac{\partial K_t^B}{\partial X^B} > 0 \Rightarrow \frac{\partial d_t^R}{\partial X^B} > 0 \quad \text{and} \quad \frac{\partial K_t^B}{\partial K_0^B} > 0 \Rightarrow \frac{\partial d_t^R}{\partial K_0^B} > 0. \tag{7}$$

Symmetric results hold for the effect of Red resources on d_t^B and d_t^R. Thus, at each point in time, a country's flow of casualties is decreasing in the resources the country commits to the war and increasing in the resources committed by its opponent.

It is worth emphasizing this result. Suppose Blue commits additional resources to the war, either via K_0^B or via X^B. First, additional Blue weapons do not imply additional Blue casualties. This is because, the military technology (Equations 1 and 2) implies that the destruction inflicted by Red on Blue is proportional to the stock of Red weapons but independent of the stock of Blue weapons (and vice versa).[3] Second, additional Blue weapons imply a higher flow of casualties for Red, which impairs Red's ability to destroy Blue weapons. The result is a lower flow of Blue casualties.

The dynamics of the flows of casualties transpire from Figure 1 and mimic the weapons stocks dynamics. When initial conditions are in the northwest quadrant, d_t^B is monotonically decreasing and d_t^R is monotonically increasing. When initial conditions are in the light-shaded area, d_t^R is monotonically increasing and d_t^B exhibits a $\cap$-shaped trajectory. Finally, when initial conditions are in the dark-shaded area, d_t^B is monotonically decreasing and d_t^R exhibits a $\cup$-shaped trajectory.

Let D_t^R and D_t^B denote total casualties at t for Red and Blue, respectively:

$$D_t^R = \int_0^t d_u^R du \text{ and } D_t^B = \int_0^t d_u^B du.$$

In the remainder of the paper I will refer to d_t^B and d_t^R as "flow-casualties" and to D_t^B and D_t^R as "casualties." When the distinction is irrelevant, I will use "casualties" as well. I show (Appendix B) that casualties can be written as

$$D_t^R = tX^R + K_0^R - K_t^R, \tag{8}$$
$$D_t^B = tX^B + K_0^B - K_t^B. \tag{9}$$

That is, a country's casualties at t are the sum of the initial weapons stock and reinforcements committed until t, net of the remaining weapons stock. Casualties at the end of war depend on how and when the war concludes.

2.3 Military conclusion

I adopt the following definition: A "military" conclusion is when a war ends because a belligerent's weapons stock reaches a critically low, exogenously determined, threshold. I assume the threshold is zero. Not all wars end with a military conclusion, but World War II is an example. The fighting ability of both the German and Japanese

[3] This technology, labeled "aimed-fire" (e.g., Taylor, 1980), can be opposed to another, "area-fire," where a country's casualties are increasing in its own weapons stock.

armed forces was close to nil by the end of the war.

The prevailing side Figure 1 illustrates the condition under which a military conclusion is a victory for Blue or for Red, or is a stalemate. Initial conditions above the stable branch imply that the Red weapons stock eventually reaches 0, while the Blue stock reaches a positive value. This can be seen from Equations (4) and (5) since $e^{t\lambda_1}$ converges to 0 while $e^{t\lambda_2}$ diverges. It follows that, when $\mathcal{B} > 0$, $\tilde{K}_t^B$ eventually increases while $\tilde{K}_t^R$ eventually decreases. Thus,

$$\begin{aligned} \mathcal{B} > 0 \quad &\Rightarrow \quad \text{Blue victory,} \\ \mathcal{B} < 0 \quad &\Rightarrow \quad \text{Red victory,} \\ \mathcal{B} = 0 \quad &\Rightarrow \quad \text{stalemate.} \end{aligned}$$

The condition for a Blue military victory can be expressed as

$$\sqrt{\theta^B} K_0^B + X^B/\sqrt{\theta^R} > \sqrt{\theta^R} K_0^R + X^R/\sqrt{\theta^B}. \tag{10}$$

I label the units in Equation (10) "efficiency units" or "fighting strength" (as in the Operations Research literature). I use these terms interchangeably. The general form for efficiency units is

$$\sqrt{\text{attrition coefficient}} \times \text{quantity of weapon.}$$

The Blue fighting strength, on the left-hand side of Equation (10), is the sum of that arising from the initial weapons stock, $\sqrt{\theta^B} K_0^B$, and that arising from reinforcements, $X^B/\sqrt{\theta^R}$. Note that the latter is also $\sqrt{\theta^R}\bar{K}^R$, so it is indeed in the same units, and that a higher $\bar{K}^R$ raises the Blue fighting strength. That is because a higher stalemate makes it harder for a country to attrit its opponent. On the right-hand side of Equation (10), the Red fighting strength is defined in the same manner.

Equation (10) is a modified version of the so-called Lanchester Square Law and deserves some comments. Consider the case where $X^B = X^R = 0$. The Blue fighting strength is then $\sqrt{\theta^B} K_0^B$, which increases faster with the weapons stocks than with the attrition coefficient. That is because an additional Blue weapon destroys Red weapons and dilutes Red's ability to attrit Blue. A higher Blue attrition coefficient serves the first purpose but not the second. (The same logic applies for Red). This property has often been viewed as a rationalization of the practice of concentrating military units while inducing the enemy to divide its military units.

In the general case, when $X^R, X^B \geq 0$, there is an additional benefit from a higher attrition coefficient: It reduces the opposing force's fighting strength arising from reinforcements. A higher Blue attrition coefficient, for instance, reduces the contribution of Red reinforcements to Red fighting strength. That is because Blue destroys Red reinforcements with a lower stalemate stock, and, thus, it is easier for Blue to exceed its stalemate and attrit the Red weapons stock.

I assume, for the remainder of this section, that $\mathcal{B} > 0$, so that Blue is poised to obtain a military victory at τ, that is

$$K^R_\tau = 0.$$

Duration of war I show (Appendix C) that

$$\frac{\partial \tau}{\partial X^B} < 0, \qquad \frac{\partial \tau}{\partial X^R} > 0, \qquad \frac{\partial \tau}{\partial K^B_0} < 0, \qquad \frac{\partial \tau}{\partial K^R_0} > 0. \tag{11}$$

The duration of war, τ, is decreasing with the initial stock of Blue weapons and with Blue reinforcements, while it is increasing with the initial stock of Red weapons and with Red reinforcements. More generally, the duration of war before a military conclusion is decreasing in the resources of the country obtaining the military victory, and increasing in the resources of the country being militarily defeated.

The logic behind this result is as follows. When Blue allocates more resources to the war, either via the initial weapons stock or via reinforcements, Blue's ability to attrit Red is heightened and the Red weapons stock depletes faster. Hence, the war is shorter. If Red allocates more resources to the war and $\mathcal{B}$ remains positive, Blue suffers additional flow-casualties. This slows down Blue's ability to attrit Red, and, therefore, the war takes longer before the Blue military victory.

It is useful to characterize the level curves of τ in the (K^R, K^B) plane. Even though initial weapon stocks are predetermined when the war starts, they are the results of prewar investments. In Section 3 I discuss the role of prewar investments for the

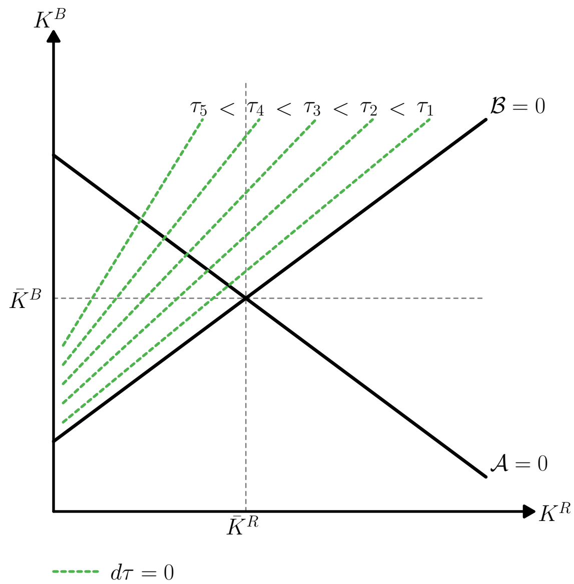


Figure 2: Level curves of the duration of war before a military conclusion when $\mathcal{B} > 0$

outcomes of war and the characterization of the level curves of τ becomes useful.

I show (Appendix C) that the level curves are straight lines with slopes

$$\left.\frac{dK_0^B}{dK_0^R}\right|_{d\tau=0} = v_1 \underbrace{\left(-\frac{\cosh(\tau\lambda_1)}{\sinh(\tau\lambda_1)}\right)}_{\text{decreasing, } \lim_{\tau\to\infty}=1}, \tag{12}$$

where cosh and sinh are the hyperbolic cosine and sine functions, respectively. It follows from (11) that the slopes of the level curves are increasing with K_0^B and decreasing with K_0^R. Figure 2 represents a set of level curves. Since duration is decreasing in K_0^B, higher curves correspond to shorter wars.

Two points deserve commenting at this stage. First, the result in Equation (12) implies that, for τ to remain constant, initial conditions have to change as

$$\sqrt{\theta^B} dK_0^B = \sqrt{\theta^B} dK_0^R \left(-\frac{\cosh(\tau\lambda_1)}{\sinh(\tau\lambda_1)}\right).$$

Suppose the change in Blue fighting strength was, instead, equal to the change in Red fighting strength, i.e., $\sqrt{\theta^B}dK_0^B = \sqrt{\theta^R}dK_0^R$. This would maintain the condition for Blue's military victory, i.e., $d\mathcal{B} = 0$. Blue, however, would need extra time to destroy the additional Red weapons and, thus, the war would be longer. To obtain victory in the same time, the increase in Red fighting strength must be met by an increase in Blue fighting strength larger than what is necessary to maintain $\mathcal{B}$ constant. Hence the second, larger-than-one term on the right-hand side of Equation (12).

Second, as the war is shorter, each additional Red weapons must be destroyed faster for the duration of war to remain the same. This is why the level curves are steeper as the duration of war decreases.

The level curves of τ in the (X^B, X^R) plane are also straight lines with slopes

$$\left.\frac{dX^B}{dX^R}\right|_{d\tau=0} = v_1 \underbrace{\frac{\sinh(\tau\lambda_1)}{1-\cosh(\tau\lambda_1)}}_{\text{decreasing, } \lim_{\tau\to\infty}=1} . \tag{13}$$

The interpretation, here, is similar to that of Equation (12). The increase in X^B needed to offset an increase in X^R and maintain duration is larger than the increase needed for $d\mathcal{B} = 0$. Furthermore, the level curves are steeper for shorter wars because additional Red reinforcements must be destroyed faster.

End-of-war stocks I show (Appendix D) that the Blue end-of-war stock is

$$K_\tau^B = \bar{K}^B + \sqrt{\left(v_1\bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}}, \tag{14}$$

and that

$$K_\tau^B > K_0^B \quad \text{whenever} \quad K_0^R < 2\bar{K}^R.$$

Thus, if the initial stock of Red weapons is low enough, Blue reinforcements deployed throughout the war exceed the destruction caused by Red, and the final Blue weapons stock is above its initial value. Panels A and B of Figure 3 represent, with light- and dark-shaded areas, the initial conditions leading to either $K_\tau^B > K_0^B$ or $K_\tau^B < K_0^B$.

How does K_τ^B depend upon resources committed before and during the war? I show

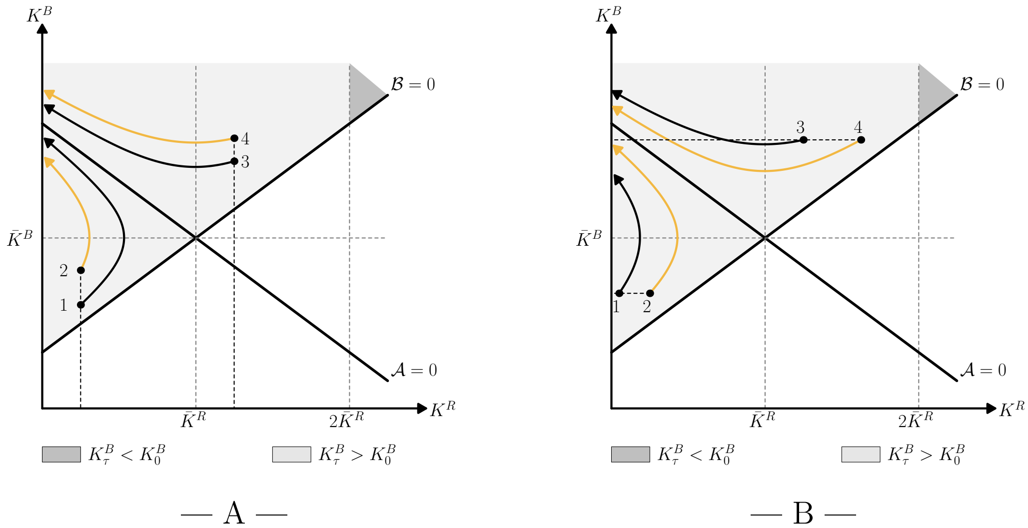


Figure 3: The determination of the Blue end-of-war weapons stock when $\mathcal{B} > 0$

(Appendix D) that

$$\frac{\partial K^B_\tau}{\partial K^B_0} = \frac{K^B_0 - \bar{K}^B}{K^B_\tau - \bar{K}^B} \lesseqqgtr 0 \quad \text{and} \quad \frac{\partial K^B_\tau}{\partial X^B} = \frac{1}{\theta^B}\frac{K^R_0}{K^B_\tau - \bar{K}^B} > 0.$$

To understand these effects, note that Equations (1) and (2) imply $\partial^2 K^B_t/\partial t^2 = \theta^R\theta^B(K^B_t - \bar{K}^B)$: The growth rate of K^B_t is decreasing below the stalemate and increasing above. Below the stalemate, for instance, Blue does not offset Red reinforcements and, hence, the Red weapons stock grows, impairing the growth of K^B_t.

When $K^B_0 < \bar{K}^B$, an increase in K^B_0 implies that the Blue weapons stock starts higher but grows at a decreasing rate until $K^B_t = \bar{K}^B$. Since the war is shorter (Equation 11) the final result is a lower end-of-war weapons stock (trajectories 1 and 2 in Panel A of Figure 3). When $K^B_0 > \bar{K}^B$, the additional K^B_0 implies that the Blue weapons stock starts higher and grows at an increasing rate, hence the higher final stock despite the shorter war (trajectories 3 and 4 in Panel A of Figure 3). Blue reinforcements raise the final weapons stock despite the shorter war, regardless of initial conditions.

I also show that

$$\frac{\partial K_\tau^B}{\partial K_0^R} = -\upsilon_1^2 \frac{K_0^R - \bar{K}^R}{K_\tau^B - \bar{K}^B} \lesseqqgtr 0 \quad \text{and} \quad \frac{\partial K_\tau^B}{\partial X^R} = \frac{1}{\theta^B} \frac{K_\tau^B - K_0^B}{K_\tau^B - \bar{K}^B} \lesseqqgtr 0.$$

If $K_0^R < \bar{K}^R$, additions to the initial Red stock do not offset Blue reinforcements, and, since the war is longer, the final Blue stock increases (trajectories 1 and 2 in Panel B of Figure 3). If, however, $K_0^R > \bar{K}^R$, the additional Red weapons offset Blue reinforcements in the early stages of the war and the final Blue stock decreases despite the longer war (trajectories 3 and 4 in Panel B of Figure 3). Finally, an increase in Red reinforcement lowers the final Blue stock if $K_\tau^B < K_0^B$, that is $K_0^R > 2\bar{K}^R$.

Casualties Casualties at τ follow from Equations (8) and (9):

$$D_\tau^R = \tau X^R + K_0^R \quad \text{and} \quad D_\tau^B = \tau X^B + K_0^B - K_\tau^B. \tag{15}$$

Red loses all the resources it commits to the war. Blue casualties are mitigated by the end-of-war Blue weapons stock given in Equation (14). I show (Appendix E) that

$$\frac{\partial D_\tau^R}{\partial X^B} < 0, \qquad \frac{\partial D_\tau^R}{\partial X^R} > 0, \qquad \frac{\partial D_\tau^R}{\partial K_0^B} < 0, \qquad \frac{\partial D_\tau^R}{\partial K_0^R} > 0, \tag{16}$$

and

$$\frac{\partial D_\tau^B}{\partial X^B} < 0, \qquad \frac{\partial D_\tau^B}{\partial X^R} > 0, \qquad \frac{\partial D_\tau^B}{\partial K_0^B} < 0, \qquad \frac{\partial D_\tau^B}{\partial K_0^R} > 0. \tag{17}$$

That is, in a Blue military victory, Blue casualties are reduced by Blue resources committed to the war and increased by Red resources. Red casualties behave in the **same** (not symmetric) manner. That is, Red casualties are also reduced by Blue resources and increased by Red resources.

To understand these results, consider the effects of an increase in the initial stock of Blue weapons, from K_0^B to $K_{0,\text{new}}^B$, illustrated in Figure 4. Panel A represents Blue flow-casualties (the solid and dashed lines) and Blue casualties (the areas under the lines). Blue flow-casualties converge to zero because the stock of Red weapons converges to zero in a Blue military victory. Recall that Blue flow-casualties are lower at each point in time when K_0^B is higher (Equation 6), implying that the dashed line is below the solid line. Thus, the light-shaded area in Panel A indicates a reduction in

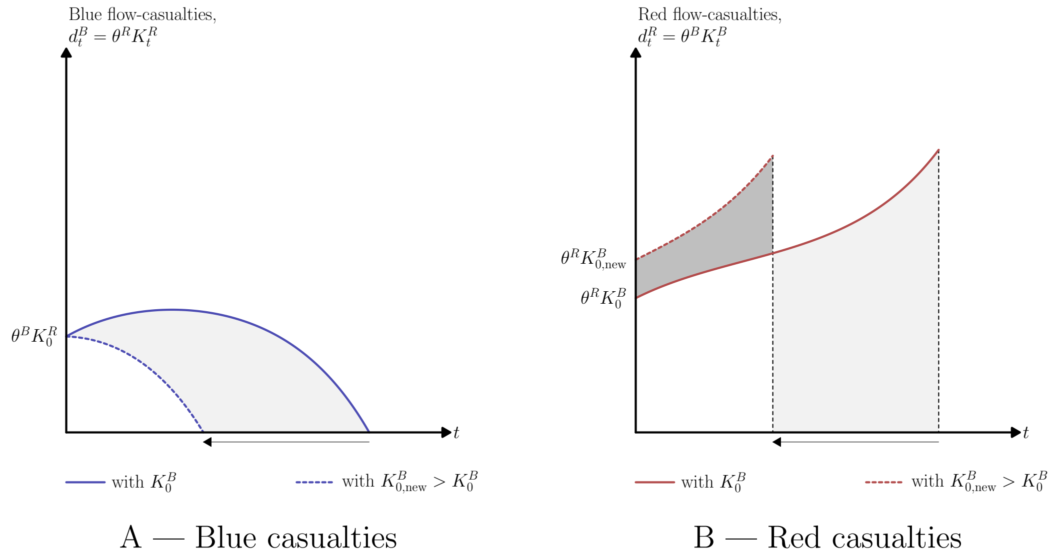


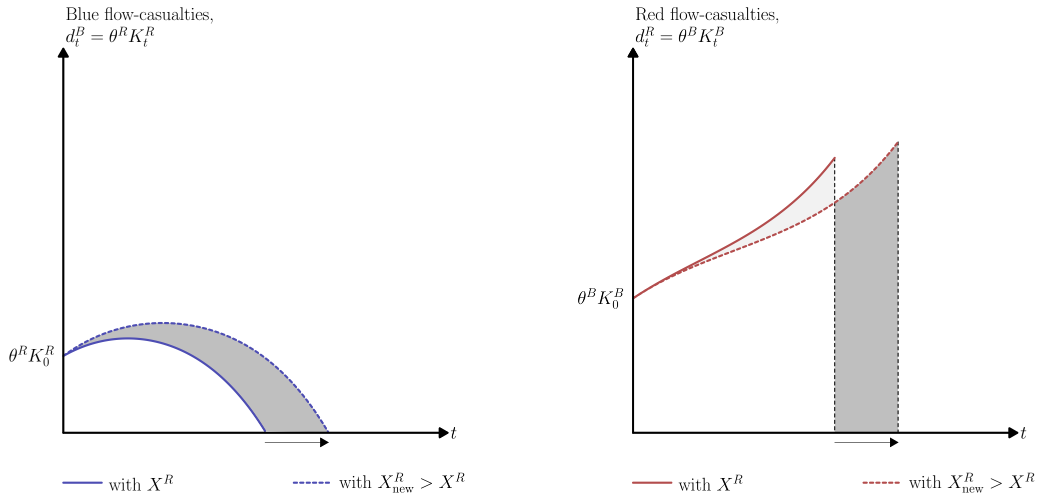
Figure 4: The effect of K_0^B on casualties when $\mathcal{B} > 0$

Note: In panel A, the vertical axis measures Blue flow-casualties, d_t^B. Thus, the area under a line and up to time t represents Blue casualties, $D_t^B = \int_0^t d_u^B du$. The left-pointing arrow under the horizontal axis indicates the shorter duration of war. Panel B reads in the same way.

Blue casualties, D_τ^B. This reduction combines two effects acting in the same direction: the lower flow-casualties at each point in time and the shorter war (Equation 11).

Panel B represents Red flow-casualties and Red casualties. Two effects operate in opposite directions. First, Red flow-casualties are higher at each point in time (Equation 7). Second, the war is shorter. The dark-shaded area represents the increase in Red casualties due to the first effect. The light-shaded area represents the decrease due to the second effect. In the end, the second effect dominates, as indicated in Equation (16). Thus, in a Blue military victory, Red casualties are reduced when the initial stock of Blue weapons is larger. The effect of an increase in Blue reinforcements, X^B can be understood in a similar way.

Figure 5 shows the effect of an increase in Red resources, namely X^R. In Panel A, Blue flow-casualties are higher at each point in time because X^R is higher—this is the symmetric effect of that described in Equation (7) for d_t^R. Thus, the dark shaded area indicates an increase in Blue casualties resulting from two effects acting in the same direction: the higher flow-casualties and the longer war. Panel B shows Red flow-



A — Blue casualties

B — Red casualties

Figure 5: The effect of X^R on casualties when $\mathcal{B} > 0$

Note: In panel A the vertical axis measures Red flow-casualties, d_t^R. Thus, the area under a line and up to time t represents Red casualties, $D_t^R = \int_0^t d_u^R du$. The right-pointing arrow under the horizontal axis indicates the longer duration of war. Panel B reads in the same way.

casualties, which are lower at each point in time—this is the symmetric effect of that described in Equation (6) for d_t^B. The effect on Red casualties combines two effects acting in opposite directions: The light-shaded area represents a reduction of Red casualties due to fewer flow-casualties. The dark-shaded area represents an increase due to the longer war. In the end, as indicated in Equation (16), the second effect dominates and Red casualties increase as a result of increased Red reinforcements. The effect of the initial Red weapons stock can be understood similarly.

The results in Equations (16) and (17) can be stated more generally as the following: At a military conclusion, the casualties on **both** sides are decreasing with the resources committed by the belligerent obtaining the military victory and increasing with the resources committed by the belligerent being militarily defeated.

2.4 Political conclusion

I assume there are exogenously-determined threshold levels of casualties beyond which a country decides to sue for peace. Let $\bar{D}^R$ and $\bar{D}^B$ denote these thresholds for Red and Blue, respectively. I label such conclusion "political," as opposed to "military," because the fighting strength of the country suing for peace need not be lower than its opponent's. For example, Blue can sue for peace when $\mathcal{B} > 0$.

The Vietnam war is an example of a conflict that did not reach a military conclusion in the sense of Section 2.3. No belligerent was militarily incapable of fighting when the war concluded. Instead, political forces in the United States compelled decision makers to reduce the U.S. involvement in the war. The peak of U.S. troops in Vietnam was in April 1969 (Anderson, 2002, p. 187), and evidence of political discontent with the war is numerous, e.g., the anti-war movement or the repeal of the Tonkin Gulf Resolution by the U.S. Senate in 1970.

World War I is another example, although more controversial. Allied forces had not marched into Germany when the war ended, and Douglas Haig, the commander of the British Expeditionary Force, said of the November 1918 armistice: "*Germany is not broken in a military sense*" (Liddell Hart, 2012, ch. 13).

Duration of war Blue sues for peace at $\tau^B < \tau$ if casualties reach their threshold value and Red has not yet sued for peace, i.e., if $D^B_{\tau^B} = \bar{D}^B$ or, using Equation (9),

$$K^B_{\tau^B} = \tau^B X^B + K^B_0 - \bar{D}^B.$$

Note that if $\tau^B > 0$ exists, it is unique because the slope of the left-hand side (with respect to time) is less than that of the right-hand side. Similarly, Red sues for peace at $\tau^R < \tau$ if Blue has not done so already and

$$K^R_{\tau^R} = \tau^R X^R + K^R_0 - \bar{D}^R.$$

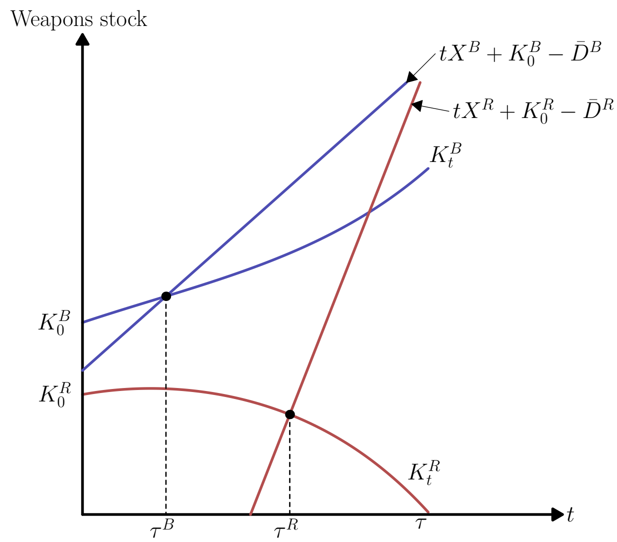


Figure 6: Weapons stock trajectories and political thresholds when $\mathcal{B} > 0$

Recall that casualties are cumulative so that D_t^R and D_t^B are monotonically increasing over time. It follows that, if $\mathcal{B} > 0$,

$$\text{the conclusion is } \begin{cases} \text{military at } \tau & \text{if } \ \bar{D}^R \geq D_\tau^R \ \text{ and } \ \bar{D}^B \geq D_\tau^B, \\ \text{political at } \min\left\{\tau^B, \tau^R\right\} & \text{otherwise,} \end{cases}$$

where D_τ^R and D_τ^B are given by Equation (15).[4] Figure 6 represents weapons stocks trajectories and the determination of τ, τ^B, and τ^R in an example where $\mathcal{B} > 0$ and Blue's political threshold, $\bar{D}^B$, is low enough that τ^B is prior to the date at which Blue would obtain a favorable military conclusion, τ, and prior to the date at which Red would sue for peace, τ^R. Thus, Blue concludes the war at τ^B by suing for peace even though it is poised to obtain a military victory.

How do resources allocated to war by either belligerent affect the date at which one

[4] If $\mathcal{B} < 0$ the same rule applies, but the date τ of a military conclusion is different from that discussed in Section 2.3 and end-of-war casualties, D_τ^B and D_τ^R, are different from those given in Equation (15).

of them initiates a political conclusion? I show (Appendix F) that

$$\frac{\partial \tau^B}{\partial X^B} > 0, \qquad \frac{\partial \tau^B}{\partial X^R} < 0, \qquad \frac{\partial \tau^B}{\partial K_0^B} > 0, \qquad \frac{\partial \tau^B}{\partial K_0^R} < 0, \tag{18}$$

and

$$\frac{\partial \tau^R}{\partial X^B} < 0, \qquad \frac{\partial \tau^R}{\partial X^R} > 0, \qquad \frac{\partial \tau^R}{\partial K_0^B} < 0, \qquad \frac{\partial \tau^R}{\partial K_0^R} > 0. \tag{19}$$

The date at which a country reaches its political threshold is increasing with the resources committed by the country and decreasing with the resources committed by its opponent. Note that these results are independent of the sign of $\mathcal{B}$. They hold regardless of which country is poised to obtain a military victory.

Suppose Blue commits additional resources to the war, either via K_0^B or via X^B. This reduces the flow of Blue casualties at each point in time (Equation 6) and, thus, lengthens the time necessary for Blue to reach its political threshold: τ^B increases. Again, this is because additional Blue weapons imply a higher flow of Red casualties (Equation 7) and, thus, impair Red's ability to destroy Blue weapons. The higher flow of Red casualties shortens the time necessary for Red to reach its political threshold: τ^R decreases. The effects of Red resources on τ^B and τ^R have similar explanations.

I show (Appendix F) that the level curves of τ^B in the (K_0^R, K_0^B) plane and the (X^R, X^B) plane are straight lines with slopes

$$\left.\frac{dK_0^B}{dK_0^R}\right|_{d\tau^B=0} = v_1 \underbrace{\frac{\sinh\left(\tau^B\lambda_1\right)}{1-\cosh\left(\tau^B\lambda_1\right)}}_{\text{decreasing, } \lim_{\tau^B\to\infty}=1}, \quad \text{and} \quad \left.\frac{dX^B}{dX^R}\right|_{d\tau^B=0} = v_1 \underbrace{\frac{1-\cosh\left(\tau^B\lambda_1\right)}{\sinh\left(\tau^B\lambda_1\right)-\tau^B\lambda_1}}_{\text{decreasing, } \lim_{\tau^B\to\infty}=1}. \tag{20}$$

As was the case for the level curves of τ, the level curves of τ^B are steeper than the stable branch (with slope v_1). That is because if Blue met increased Red resources with an equal increase in fighting strength, then Blue would experience higher casualties and reach its political threshold earlier. The level curves of τ^B are also steeper the shorter the war. That is because, the shorter the war, the more resources Blue must deploy to avoid the fast accumulation of flow-casualties.

Casualties The casualties of the country suing for peace are given by its political threshold. How are its opponents' casualties affected by resources? I show (Appendix

G) that

$$\frac{\partial D^R_{\tau^B}}{\partial X^B} > 0, \qquad \frac{\partial D^R_{\tau^B}}{\partial X^R} < 0, \qquad \frac{\partial D^R_{\tau^B}}{\partial K^B_0} > 0, \qquad \frac{\partial D^R_{\tau^B}}{\partial k^R_0} < 0,$$

and

$$\frac{\partial D^B_{\tau^R}}{\partial X^B} < 0, \qquad \frac{\partial D^B_{\tau^R}}{\partial X^R} > 0, \qquad \frac{\partial D^B_{\tau^R}}{\partial K^B_0} < 0, \qquad \frac{\partial D^B_{\tau^R}}{\partial k^R_0} > 0.$$

At a political conclusion initiated by Blue, at date τ^B, Red casualties are increasing in Blue resources and decreasing with Red resources. Symmetrically, if Red initiates the political conclusion at date τ^R, Blue casualties are increasing in Red resources and decreasing with Blue resources.

Consider a conclusion initiated by Blue at τ^B. When Blue commits additional resources to the war, either via K^B_0 or via X^B, two effects raise Red casualties: First, Red flow-casualties are higher at each point in time (Equation 7). Second, the war is longer because the date τ^B is pushed into the future (Equation 18). If Red commits additional resources to the war, two effects lower Red casualties: First Red flow-casualties are lower—this is the symmetric effect of that described in Equation (6) for d^B_t. Second the war is shorter (Equation 18). A similar explanation applies to the effects of resources on Blue casualties when Red initiates a political conclusion at τ^R.

3 Using the model

3.1 Rich v. poor countries

In December 1941 Japan declared war on the United States even though the latter was a larger economy: The gross domestic product (GDP) difference was 5-fold and the population difference was almost 2-fold, both in favor of the United States.[5] In February 2022 Russia invaded Ukraine. The latter was significantly smaller: The GDP difference was 8-fold and the population difference was more than 3-fold, both in favor of Russia.[6] How are differences in GDP relevant for the outcomes of war?

I reinterpret the model in terms of GDP, prewar saving rates, and wartime saving rates. Let Y^R and Y^B denote the (constant) GDP of Red and Blue, respectively. I

[5] Maddison Project Database 2020.

[6] 2018 figures from the Maddison Project Database 2020.

assume that, before the war, Red and Blue are at steady states where they allocate constant fractions of GDP to their military. Thus, at the start of war their initial weapons stocks are proportional to their GDP:

$$K_0^B = s^B Y^B \quad \text{and} \quad K_0^R = s^R Y^R.$$

I refer to s^R and s^B as prewar (military) saving rates for the sake of exposition.[7] I further assume that, in wartime, Red and Blue allocate constant fractions (possibly different than in peacetime) of their GDP to reinforcements

$$X^B = \sigma^B Y^B \quad \text{and} \quad X^R = \sigma^R Y^R,$$

where σ^B and σ^R are wartime investment rates in weapons.

The prevailing side The condition for a Blue military victory in Equation (10) can be written $\mathcal{B}/Y^B > 0$, yielding

$$s^B + \frac{\sigma^B}{\sqrt{\theta^R \theta^B}} > \underbrace{\frac{\sqrt{\theta^R} Y^R}{\sqrt{\theta^B} Y^B}}_{\mathcal{Y}} \left(s^R + \frac{\sigma^R}{\sqrt{\theta^R \theta^B}} \right). \tag{21}$$

I refer to $\sqrt{\theta^B} Y^B$ as the fighting strength of Blue GDP. That is, if Blue converted the entirety of its GDP into weapons, the fighting strength it would obtain is precisely $\sqrt{\theta^B} Y^B$. I refer to $\sqrt{\theta^R} Y^R$ similarly.

The ratio $\mathcal{Y}$ of Red-to-Blue fighting strength of GDP, in Equation (21), indicates the role of each country's GDP in determining the prevailing side in a military conclusion. Suppose, for instance, that $s^R > s^B$ and $\sigma^R > \sigma^B$. That is, Red allocates a larger fraction of its GDP to the military both in peacetime and in wartime. This does not imply that Red wins the war since Blue could be a "bigger" country. But how exactly does the size of GDP matter? The answer is through $\mathcal{Y}$. With $\mathcal{Y}$ small enough, the condition for a Blue military victory is satisfied.

Note that, as in the discussion of condition (10), that the role of military technology,

[7] With a constant saving rate, as in the Solow model for instance, the capital-to-output ratio is proportional to the saving rate, albeit not equal to it.

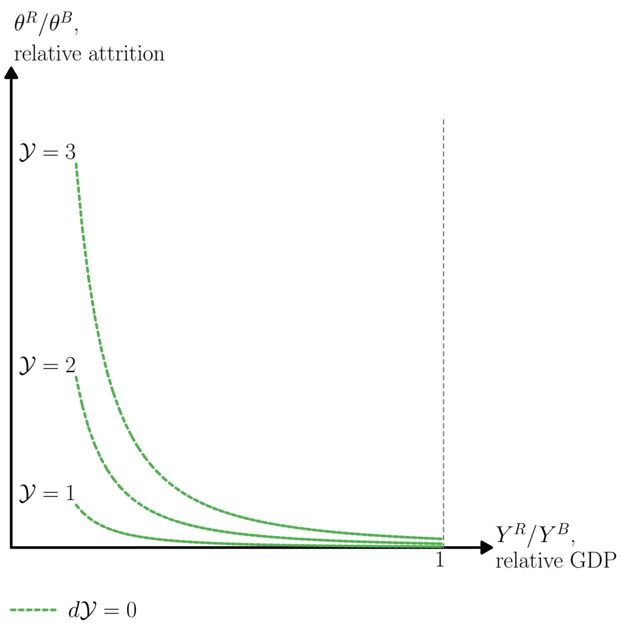


Figure 7: Level curves of relative fighting strength

via the attrition coefficients, is small relative to the role of GDP. That is, the elasticity of $\mathcal{Y}$ with respect to relative technology θ^R/θ^B is 1/2, while the elasticity of $\mathcal{Y}$ with respect to relative GDP is 1. This is illustrated in Figure 7, which shows level curves of $\mathcal{Y}$ in the $(Y^R/Y^B, \theta^R/\theta^B)$ plane. As Red GDP becomes smaller relative to Blue GDP, the relative technology advantage in favor of Red, necessary to maintain $\mathcal{Y}$, increases quadratically. A poor country is thus at a disadvantage relative to a rich country not only because it has fewer resources, but also because the resource gap is increasingly difficult to offset with better technology the poorer the country is.

Equation (21) gives meaning to the notion of a country "outproducing" its opponent. Recall that s^R and s^B are pre-determined when the war breaks out. Let $\hat{\sigma}^B$ denote the solution σ^B of Equation (21) at equality, assuming $\sigma^R = 1$. That is, $\hat{\sigma}^B = \mathcal{Y}(1 + s^R\sqrt{\theta^R\theta^B}) - s^B\sqrt{\theta^R\theta^B}$. Then, any Blue wartime investment rate $\sigma^B > \hat{\sigma}^B$ implies $\mathcal{B} > 0$ for any Red wartime investment rate less than 100%. In other words, Blue "outproduces" Red. Without importing reinforcements, Red cannot obtain a military victory. The lower $\mathcal{Y}$, the easier it becomes for Blue to outproduce Red

The duration of war Assume that the condition for a Blue military victory is satisfied (Equation 21) and that $K^R_\tau = 0$. The role of GDP for the duration of war can be gauged by deriving, from Equations (12) and (13), that

$$\left.\frac{ds^B}{ds^R}\right|_{d\tau=0} = \mathcal{Y}\ \underbrace{\left(-\frac{\cosh(\tau\lambda_1)}{\sinh(\tau\lambda_1)}\right)}_{\text{decreasing, } \lim_{\tau\to\infty}=1} \quad \text{and} \quad \left.\frac{d\sigma^B}{d\sigma^R}\right|_{d\tau=0} = \mathcal{Y}\ \underbrace{\frac{\sinh(\tau\lambda_1)}{1-\cosh(\tau\lambda_1)}}_{\text{decreasing, } \lim_{\tau\to\infty}=1} .$$

These equations describe the level curves of τ in the pre- and wartime saving rates planes. Consider increases in Red's pre- and/or wartime investment, s^R and/or σ^R. Since Blue is poised to obtain a military victory, these increases lengthen the war (Equation 11). Blue can offset these effects by raising s^B and/or σ^B. Note that the lower $\mathcal{Y}$, the cheaper it is for Blue to do so.

Blue may reach its political threshold before it obtains the military victory, however. Furthermore, Red can hasten such political conclusion by raising s^R and/or σ^R and thus committing additional resources to the war (Equation 18). Equations (20) imply

$$\left.\frac{ds^B}{ds^R}\right|_{d\tau^B=0} = \mathcal{Y}\ \underbrace{\frac{\sinh\left(\tau^B\lambda_1\right)}{1-\cosh\left(\tau^B\lambda_1\right)}}_{\text{decreasing, } \lim_{\tau\to\infty}=1} \quad \text{and} \quad \left.\frac{d\sigma^B}{d\sigma^R}\right|_{d\tau^B=0} = \mathcal{Y}\ \underbrace{\frac{1-\cosh\left(\tau^B\lambda_1\right)}{\sinh\left(\tau^B\lambda_1\right)-\tau^B\lambda_1}}_{\text{decreasing, } \lim_{\tau\to\infty}=1} .$$

Thus, Blue can offset the effects of s^R and/or σ^R on τ^B by raising s^B and/or σ^B in turn. As above, the lower $\mathcal{Y}$, the cheaper it is for Blue to do so.

It is interesting to discuss these results. Consider the Pacific war. Historical evidence (e.g., Nolan, 2017, Toll, 2012) indicates that the Japanese military expected a short war because they assumed the United States would quickly sue for peace. The results above emphasize the weaknesses in this reasoning. Even though the Japanese military was large in 1941 while the U.S. military was not, this was not sufficient for Japan to prevail militarily. Indeed, discussions of the Pacific war often emphasize the role of U.S. economic dominance: Despite a low initial weapons stock, the U.S. outproduced Japan and prevailed. Although operating in the context of the model, this mechanism alone was, again, not sufficient. The economic dominance of the U.S. also limited casualties so that the U.S. did not need to sue for peace before prevailing militarily.

3.2 Waiting

During World War II Germany declared war on the United States in December 1941. Yet, the first major operation conducted by the United States against German forces was initiated almost a year later, with the invasion of French North Africa in November 1942. That was because the United States had the opportunity to wait before engaging the fight, and used it. In contrast, in the Pacific theater, the United States had to react to a variety of situations caused by Japanese expansion throughout 1942: in the Coral Sea in May, at Midway in June, and at Guadalcanal in August. The opportunity to wait was not available. In the Western European theater of World War II, the Phoney War is another example of a period (about 8 months) during which little fighting took place. How does the opportunity to wait, or lack thereof, matter for the outcomes of war?

Suppose that Red and Blue have declared war at time 0 and that $\mathcal{B}/Y^B > 0$, such that Blue is poised for a military victory at date τ. Suppose there is no fighting for an interval of time of length n. During that time reinforcements act as investment flows. They accumulate into weapons stocks, which are not being depleted, and initial conditions change according to

$$dK_0^B = nX^B = n\sigma^B Y^B \quad \text{and} \quad dK_0^R = nX^R = n\sigma^R Y^R.$$

The prevailing side The change in initial conditions caused by n implies

$$d(\mathcal{B}/Y^B) = n\left(\sigma^B - \mathcal{Y}\sigma^R\right).$$

Thus, if the wartime saving rate of Blue is large enough relative to that of Red, that is, if $\sigma^B > \mathcal{Y}\sigma^R$, then $d(\mathcal{B}/Y^B) > 0$ and the condition for a military victory remains satisfied after any waiting period n. If $\sigma^B < \mathcal{Y}\sigma^R$, however, $d(\mathcal{B}/Y^B) < 0$ and a delay beyond the threshold

$$n^* = \frac{\mathcal{B}/Y^B}{\mathcal{Y}\sigma^R - \sigma^B}$$

leads to a Red military victory:

$$
\begin{aligned}
\sigma^B \geq \mathcal{Y}\sigma^R \quad &\Rightarrow \quad \text{Blue victory after any delay } n, \\
\sigma^B < \mathcal{Y}\sigma^R \quad &\Rightarrow \quad \text{Blue victory after any delay } n < n^*, \\
&\Rightarrow \quad \text{Red victory after any delay } n > n^*, \\
&\Rightarrow \quad \text{stalemate after a delay } n = n^*.
\end{aligned}
$$

The duration of war Suppose that n is such that Blue remains poised to obtain a military victory at τ, that is, $K_\tau^R = 0$. Equation (12) implies

$$
d\tau < 0 \Leftrightarrow \sigma^B > \sigma^R \mathcal{Y} \underbrace{\left(-\frac{\cosh(\tau\lambda_1)}{\sinh(\tau\lambda_1)}\right)}_{\text{decreasing, } \lim_{\tau\to\infty}=1} .
$$

When σ^B is sufficiently large with respect to σ^R, waiting reduces the duration of the *fighting*.[8] Note the role of $\mathcal{Y}$. The threshold σ^B at which Blue can reduce the duration of fighting via a waiting strategy is lower when $\mathcal{Y}$ is lower. In other words, if Blue is rich enough, it is cheaper for Blue to reduce the duration of the fighting—cheaper in the sense of being achievable via lower wartime investment.

Equation (20) implies

$$
d\tau^B > 0 \Leftrightarrow \sigma^B > \sigma^R \mathcal{Y} \underbrace{\frac{\sinh\left(\tau^B\lambda_1\right)}{1-\cosh\left(\tau^B\lambda_1\right)}}_{\text{decreasing, } \lim_{\tau^B\to\infty}=1} .
$$

A large enough σ^B implies lower Blue flow-casualties and, thus, pushes the date at which Blue would sue for peace for political reasons further into the future. Note again the role of $\mathcal{Y}$: The lower the relative fighting strength of Red GDP, the lower the threshold value of σ^B. In other words, it is cheaper for a rich country to avoid its political threshold and having to sue for peace.

[8] The duration of the war itself changes from τ to $\tau + n + d\tau$. Thus, the war is shorter if $d\tau < -n$.

3.3 Foreign support

In the Russia-Ukraine war started in February 2022, Ukraine is assessed by the international community to be unable to withstand the Russian military on its own. This assessment prompted the United States and other countries to provide military support to Ukraine. How does a third party support matter for the outcomes of war?

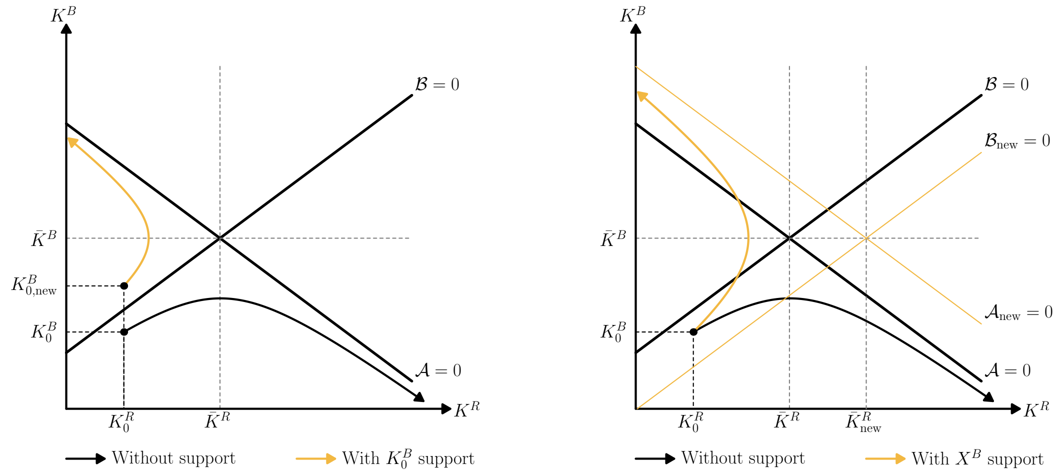
Consider a conflict where Red is poised for a military victory, i.e., $\mathcal{B} < 0$. Assume that a third party, e.g., a coalition of foreign countries, supports Blue with a one-time transfer of weapons, S_K, at date 0 and/or a commitment to a flow of reinforcements S_X at each point in time. Let $K^B_{0,\text{new}} = K^B_0 + S_K$ and $X^B_{\text{new}} = X^B + S_X$ denote the new level of Blue's initial weapons stock and reinforcements, respectively. Define $\bar{K}^R_{\text{new}}$ and $\mathcal{B}_{\text{new}}$ accordingly, as in Sections 2.1 and 2.2.

The prevailing side The condition for a Blue military victory with foreign support is $\mathcal{B}_{\text{new}}/Y^B > 0$, that is,

$$s^B + S_K/Y^B + \frac{\sigma^B + S_X/Y^B}{\sqrt{\theta^R\theta^B}} > \mathcal{Y}\left(s^R + \frac{\sigma^R}{\sqrt{\theta^R\theta^B}}\right). \tag{22}$$

The effect of a time-0 transfer of weapons, that is, $S_K > 0$ and $S_X = 0$, is represented in Panel A of Figure 8. The initial condition is below the stable branch and thus, Blue is on a trajectory to a military defeat (black). If S_K is large enough, the initial condition is above the stable branch and Blue obtains a military victory (orange).

The effect of a commitment to reinforcements at each point in time, that is $S_K = 0$ and $S_X > 0$, is represented on Panel B. Additional reinforcements raise the stalemate value of Red's weapons stock to $\bar{K}^R_{\text{new}} = X^B_{\text{new}}/\theta^R > \bar{K}^R$, making it harder for Red to attrit the Blue force. Graphically, the stable and unstable branches translate to the right while the initial condition (K^R_0, K^B_0) does not change. Instead of being below the stable branch, i.e., $\mathcal{B} < 0$, the initial condition is above the new stable branch, i.e., $\mathcal{B}_{\text{new}} > 0$. The without-support (black) arrow represents the dynamics of war absent foreign support: Blue's weapons stock goes to zero. The with-support (orange) arrow represents the dynamics with foreign support: Red's weapons stock goes to zero.



A — Time-0 support
$S_K > 0$ and $S_X = 0$

B — Reinforcement support
$S_K = 0$ and $S_X > 0$

Figure 8: The effect of foreign support to Blue

The cost of support What is the cost-minimizing support (from the point of view of the third party) that ensures a Blue military victory? If the support is a one-time transfer of weapons, then the answer to this question is trivial: It is $S_{K,\min}$ such that Equation (22) is satisfied at equality with $S_K = S_{K,\min}$ and $S_X = 0$. With such support the system is on the stable branch and the war in a stalemate: Its duration is infinity. Any support marginally above $S_{K,\min}$ would permit a Blue military victory. Thus, it is trivial that the cost-minimizing one-time transfer maximizes the duration of war before a military conclusion and increases the likelihood of a political conclusion.

If support is a commitment to reinforcements at each point in time, $S_X > 0$, the cost of support at a military conclusion for the foreign coalition is τS_X. Recall from Equation (11) that raising Blue reinforcements lowers the duration of war, τ. Thus, higher support has two opposite effects on the cost to the foreign coalition: a direct effect increasing τS_X via S_X, and an indirect effect reducing τS_X via τ. Figure 9 illustrates the trade off. A low support (orange) that would move the stable branch such that the initial condition is just above the stable branch implies a long war. A high support (purple) implies a shorter war. The cost-minimizing support for a

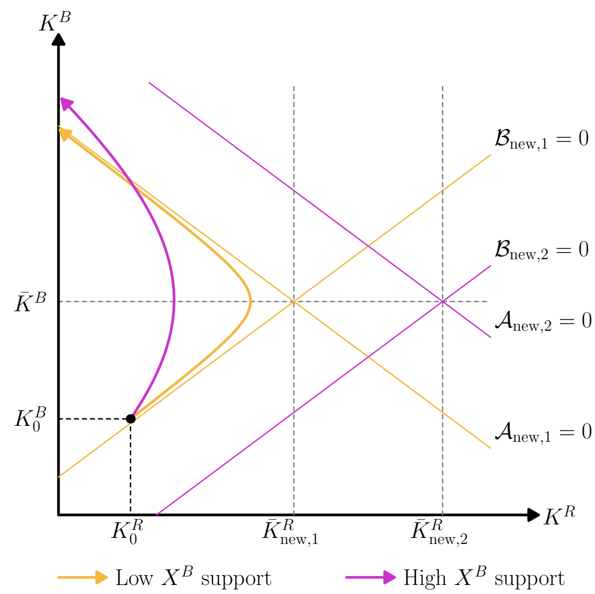


Figure 9: Foreign support trade off

military victory must satisfy

$$\frac{\partial \tau}{\partial X^B} \frac{S_X}{\tau} = -1.$$

Equations (18) and (19) indicate that foreign support to Blue pushes the date at which Blue would seek a political conclusion into the future, while bringing the date at which Red would do so closer to date 0 (the present).

4 Empirical application: Iwo Jima

In this section, I present an empirical application of the model to the case of a battle for lack of data about an entire war. My goal is to replicate the analysis of the battle of Iwo Jima, first presented by Engel (1954), and to argue that the Lanchester model provides an empirically relevant description of the process of military attrition.

Iwo Jima is a small volcanic island in the Western Pacific Ocean, almost half-way between Tokyo and Guam in the Mariana archipelago. During World War II, Iwo

Jima was deemed a strategic objective by the United States and assaulted on February 19, 1945 (D-day). Organized resistance by Japanese troops ceased on March 26.

Let Blue represent the United States and Red represent Japan. Since the existing data pertain to the number of troops, let K_t^B and K_t^R indicate the stock of U.S. and Japanese troops fighting on day t, respectively. Reinforcements, that is, X_t^B and X_t^R, are then the flow of troops landing on day t.

The U.S. data consist of the number of casualties (killed, wounded, or missing) per day (Morehouse, 1946) and number of newly landed troops per day. One set of estimates for newly landed troops is from Engel (1954) who reports a total of $73,000$ troops coming ashore: $54,000$ on D-day, $6,000$ on D-day+2, and $13,000$ on D-day+5. Samz (1972) argues that Engel's pattern of troop landing is too high and proposes alternative figures with a total of $71,245$ troops landing between D-day and D-day+10. In what follows, I consider both Engel's and Samz's estimates and construct K_t^B, the stock of fighting U.S. troops on Iwo Jima, via

$$K_{t+1}^B = K_t^B - \text{Casualites}_t + X_t^B,$$

with the initial condition $K_0^B = 0$ (no U.S. troops on the island at the start of D-day) and X_t^B given by either the Engel or the Samz patterns of reinforcements. Figure 10 shows K_t^B under each scenario.

There are no available data for the stock of Japanese fighting troops per day or Japanese casualties per day. The existing record indicates, however, that there was a stock $K_0^R = 21,500$ of Japanese troops on Iwo Jima on the eve of the U.S. invasion. Both Engel and Samz use this figure. The Japanese garrison received no reinforcements throughout the battle, i.e., $X_t^R = 0$. Both Engel and Samz estimate that there remained no Japanese troops fighting at the end of the battle, that is, $K_{35}^R = 0$.[9]

A discrete-time version of Equations (1) and (2) with time-varying reinforcements is

$$K_{t+1}^i - K_t^i = -\theta^{-i} K_t^{-i} + X_t^i. \tag{23}$$

where $(i, -i)$ stands for either (B, R) or (R, B). Engel (1954) proposed a technique

[9] Recently, Toll (2020, p. 516) reports a garrison of "about $22,000$" Japanese on the island at the start of the battle and that, except for a few hundred taken prisoner, the entire garrison was killed.

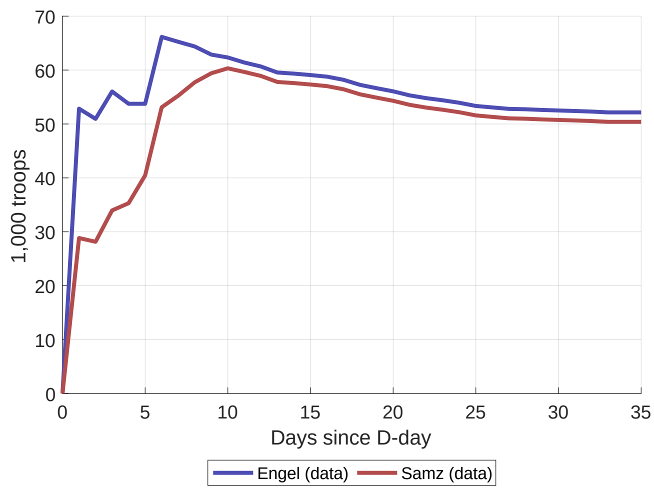


Figure 10: Two estimates of U.S. troops (K_t^B) on Iwo Jima, Feb. 19 to Mar. 26, 1945

for estimating attrition coefficients using the existing data. In the discrete version of the model presented here, Engel's technique can be described as follows: Summing over t, Equation (23) for $(i, -i) = (R, B)$ implies

$$K_{35}^R - K_0^R = -\theta^B \sum_{t=0}^{35} K_t^B + \sum_{t=0}^{35} X_t^R. \tag{24}$$

The left-hand side is observed Japanese casualties, $21,500$; the right-hand side is the sum of Japanese reinforcements, which is zero; and, finally, $\sum_{t=0}^{35} K_t^B$ can be deduced from the record of U.S. casualties and reinforcement. Under Engel's scenario for U.S. reinforcements, this figure is $1,971,820$. It follows that an estimate of the U.S. attrition coefficient is $\hat{\theta}^B = 0.011$ with Engel's data. A similar calculation with Samz's data yields $\hat{\theta}^B = 0.012$.

Given $\hat{\theta}^B$ and K_t^B, Equation (23) yields an estimate of the stock of Japanese troops per day: $\hat{K}_t^R$. The equivalent of Equation (24) for U.S. casualties is then

$$K_{35}^B - K_0^B = -\theta^R \sum_{t=0}^{35} \hat{K}_t^R + \sum_{t=0}^{35} X_t^B.$$

A – U.S. troops

B – Japanese troops and
U.S. and Japanese casualties

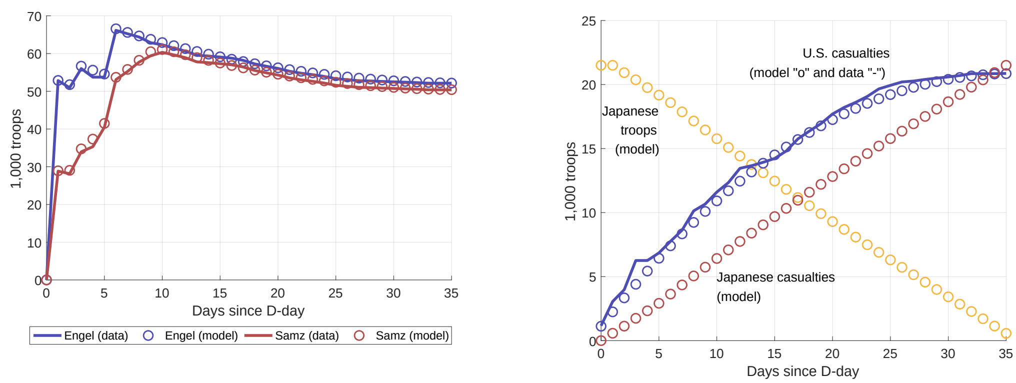


Figure 11: The Lanchester model's implications for the battle of Iwo Jima

Note: In panel A, the stock of active U.S. troops is under either Engel's assumption for U.S. reinforcement or Samz's.
Source: Author's calculations, Morehouse (1946), Engel (1954), and Samz (1972).

The left-hand side and the sum of U.S. reinforcements on the right-hand side are obtained from Morehouse (1946): $52,150$ and $73,000$, respectively. The simulated $\hat{K}_t^R$ yields $\sum_{t=0}^{35} \hat{K}_t^R = 39,436$. It follows that an estimate for the Japanese attrition coefficient is $\hat{\theta}^R = 0.052$. A similar calculation with Samz's data yields $\hat{\theta}^R = 0.050$.

Panel A of Figure 11 shows the stock of fighting U.S. troops in both the model and the data. Recall that $(\hat{\theta}^B, \hat{\theta}^R)$ is not a minimum-distance estimator, i.e., the estimates were not constructed to fit the time series K_t^B in Panel A of Figure 11. Yet, the estimated parameters yield a close fit between the observed and predicted stock of fighting U.S. troops throughout the 36 days of battle. This observation, first made by Engel (1954), indicates that the Lanchester model is a quantitatively relevant model to study attrition patterns during battles and/or wars.

Panel B of Figure 11 shows the model's implications for U.S. casualties and the unobserved Japanese troops and casualties. The model fits the daily accumulation of U.S. casualties closely, as it does for the stock of remaining U.S. troops on the island. Again, this is not the result of fitting the observed time series but is an indication of the model's empirical relevance.

Observe that the estimated stock of Japanese fighting troops is decaying monotonically because the Japanese did not receive reinforcements during the battle. Observe also that the accumulated U.S. casualties were higher than that of the Japanese and ended at nearly 21,000 in both the model and Morehouse's data. More recent work (e.g. O'Brien, 2015, p. 452) reports higher figures (more than 26,000) for U.S. casualties, making Iwo Jima one of the few battles of World War II where U.S. casualties exceeded those of the opposing force. For consistency's sake, however, I could not use the more recent estimate of total U.S. casualties without a description of the daily flow, as in Morehouse (1946).

Iwo Jima illustrates the notion of attrition well: Despite their higher attrition coefficient and thus higher ability to inflict casualties, the Japanese could not withstand the mass of U.S. military resources they faced on Iwo Jima.

5 Conclusion

Despite the large literature on war finances, there is little or no work studying how war-related expenditures affect the outcomes of war which, I argue, are of interest to economists. My goal in this paper is to suggest how to fill this gap.

Historians have argued that wars are often decided by attrition instead of "decisive" battles and genius-like generalship. Attrition, in turn, emphasizes the importance of resources in determining the outcomes of war. Hence, I use a model of resource attrition derived from combat models à la Lanchester (1916) to represent war.

I consider military conclusions, where one side cannot fight anymore for lack of resources, and political conclusions, where one side does not fight anymore for lack of political will. Under each scenario, I describe how resources determine the duration of the war, the destruction and casualties, and the prevailing side.

Some interesting results are as follows: First, a country obtaining a military victory can shorten the war by allocating more resources to it and, thus, can reduce destruction and casualties for both sides. Second, higher GDP makes the condition for a military victory more favorable and the condition for suing for peace on political grounds less favorable. Finally, there is a well-defined cost-minimizing level of support from a third-party to a small country fighting a war against a larger country.

Resources are exogenous in the model I presented. Endogenizing production (e.g. à la Solow) and the destruction of productive capacities seems a natural extension of this work, which I leave to future research. The modeling of decisions, such as the allocation of resources toward the production of consumption goods versus the production of military equipment also seems a natural extension. Finally, the collection of data and the testing of the model on an actual war instead of a battle (as in Section 4) is yet another avenue for future work.

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A Continuous time dynamic system

Consider a dynamic system of the form

$$dx/dt = \mathcal{M}x, \tag{A.1}$$

where $x \in \mathbb{R}^k$ and $\mathcal{M} \in \mathbb{R}^{k\times k}$, i.e., $\mathcal{M}$ is a $k \times k$ matrix. To find the solution of (A.1) it is useful to establish the definition of a "matrix exponential."

Definition Define the operator $\text{diag} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ as

$$\text{diag}\left(\{z_1, \ldots, z_n\}\right) = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_n \end{pmatrix},$$

that is, the operator maps a vector $z \in \mathbb{R}^n$ to a $n \times n$ matrix with the zs on its main diagonal and zeros elsewhere. Consider any scalar $a \in \mathbb{R}$ and any two vectors $x, x' \in \mathbb{R}^k$. Then

$$\begin{aligned} a\text{diag}(x) &= \text{diag}(ax), \\ \text{diag}(x) + \text{diag}(x') &= \text{diag}(x + x'), \\ \text{diag}(x)\text{diag}(x) &= \text{diag}(x^2). \end{aligned}$$

The last line implies $\text{diag}(x)^m = \text{diag}(x^m)$ for any integer m. The matrix exponential of $\mathcal{M}$ is

$$\exp(t\mathcal{M}) \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathcal{M}^n. \tag{A.2}$$

Let $\lambda \in \mathbb{R}^k = \{\lambda_1, \ldots, \lambda_k\}$ denote the eigenvalues of $\mathcal{M}$ and let $\mathcal{P} \in \mathbb{R}^{k\times k} = [v_1, \ldots, v_k]$ be the matrix of $(k\times 1)$ eigenvectors where $v_i \in \mathbb{R}^k$ is the eigenvector associated with λ_i, that is $\mathcal{M}v_i = \lambda_i v_i$. Recall that $\mathcal{M} = \mathcal{P}\text{diag}(\lambda)\mathcal{P}^{-1}$. It follows that

$$\begin{aligned} \exp(t\mathcal{M}) &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathcal{P}\text{diag}(\{\lambda_1^n, \ldots, \lambda_k^n\})\mathcal{P}^{-1}, \\ &= \mathcal{P}\text{diag}\left(\left\{\sum_{n=0}^{\infty} \frac{t^n}{n!}\lambda_1^n, \ldots, \sum_{n=0}^{\infty} \frac{t^n}{n!}\lambda_k^n\right\}\right)\mathcal{P}^{-1}. \end{aligned}$$

Using the definition of the exponential function, this is

$$\exp(t\mathcal{M}) = \mathcal{P}\text{diag}\left(\left\{e^{t\lambda_1}, \ldots, e^{t\lambda_k}\right\}\right)\mathcal{P}^{-1}. \tag{A.3}$$

Note that $\exp(0\mathcal{M}) = \text{diag}(\{1, \ldots, 1\})$.

Solution The solution of Equation (A.1) is $x_t = \exp(t\mathcal{M})x_0$, where x_0 is the initial condition. Using the definition (A.2), the solution can be verified as follows. Write

$$\frac{dx}{dt} = \frac{d}{dt}\left[\frac{t^0}{0!}\mathcal{M}^0 + \sum_{n=1}^{\infty} \frac{t^n}{n!}\mathcal{M}^n\right]x_0,$$

and note that the first term inside the brackets is independent of t. Thus,

$$\frac{dx}{dt} = \sum_{n=1}^{\infty} \frac{d}{dt} \frac{t^n}{n!} \mathcal{M}^n x_0 = \mathcal{M} \sum_{n=1}^{\infty} \frac{t^{(n-1)}}{(n-1)!} \mathcal{M}^{n-1} x_0.$$

Apply a change of variable, $j = n - 1$, then

$$\frac{dx}{dt} = \mathcal{M} \sum_{j=0}^{\infty} \frac{t^j}{j!} \mathcal{M}^j x_0 = \mathcal{M} \exp(t\mathcal{M}) x_0 = \mathcal{M} x_t.$$

Thus, using expression (A.3), the solution of (A.1) is

$$x_t = \mathcal{P} \mathrm{diag}\left(\left\{e^{t\lambda_1}, \ldots, e^{t\lambda_k}\right\}\right) \mathcal{P}^{-1} x_0. \tag{A.4}$$

B Solution of Lanchester model

Dynamics of weapons stocks Applying solution (A.4) to Equation (3) yields

$$\begin{aligned}
\begin{pmatrix} \tilde{K}_t^R \\ \tilde{K}_t^B \end{pmatrix} &= \mathcal{P} \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} \mathcal{P}^{-1} \begin{pmatrix} \tilde{K}_0^R \\ \tilde{K}_0^B \end{pmatrix}, \\
&= \frac{1}{|\mathcal{P}|} \begin{pmatrix} 1 & 1 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} \begin{pmatrix} v_2 & -1 \\ -v_1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{K}_0^R \\ \tilde{K}_0^B \end{pmatrix}, \\
&= \begin{pmatrix} e^{t\lambda_1} & e^{t\lambda_2} \\ v_1 e^{t\lambda_1} & v_2 e^{t\lambda_2} \end{pmatrix} \begin{pmatrix} (v_2 \tilde{K}_0^R - \tilde{K}_0^B)/(v_2 - v_1) \\ (\tilde{K}_0^B - v_1 \tilde{K}_0^R)/(v_2 - v_1). \end{pmatrix}.
\end{aligned}$$

Recall that $v_2 = -v_1$. The system can then be written as

$$\tilde{K}_t^R = \frac{1}{2} \left[e^{t\lambda_1} \mathcal{A} - e^{t\lambda_2} \mathcal{B}\right] \frac{1}{v_1}, \tag{B.1}$$

$$\tilde{K}_t^B = \frac{1}{2} \left[e^{t\lambda_1} \mathcal{A} + e^{t\lambda_2} \mathcal{B}\right], \tag{B.2}$$

where $\mathcal{A} = \tilde{K}_0^B + v_1 \tilde{K}_0^R$ and $\mathcal{B} = \tilde{K}_0^B - v_1 \tilde{K}_0^R$.

The effect of resources on weapons stocks dynamics It is useful to establish the following partial derivatives

$$\frac{\partial \mathcal{A}}{\partial K_0^B} = 1, \qquad \frac{\partial \mathcal{A}}{\partial X^B} = -\frac{v_1}{\theta^R}, \qquad \frac{\partial \mathcal{A}}{\partial K_0^R} = v_1, \qquad \frac{\partial \mathcal{A}}{\partial X^R} = -\frac{1}{\theta^B},$$

and

$$\frac{\partial \mathcal{B}}{\partial K_0^B} = 1, \qquad \frac{\partial \mathcal{B}}{\partial X^B} = \frac{v_1}{\theta^R}, \qquad \frac{\partial \mathcal{B}}{\partial K_0^R} = -v_1, \qquad \frac{\partial \mathcal{B}}{\partial X^R} = -\frac{1}{\theta^B}.$$

The derivatives of K_t^B are

$$\begin{aligned}
\frac{\partial K_t^B}{\partial X^B} &= \frac{1}{2}\left(e^{t\lambda_1}\left(-\frac{v_1}{\theta^R}\right)+e^{-t\lambda_1}\left(\frac{v_1}{\theta^R}\right)\right)=\frac{\sinh(t\lambda_1)}{\lambda_1}>0, &\text{(B.3)}\\
\frac{\partial K_t^B}{\partial X^R} &= \frac{1}{\theta^B}+\frac{1}{2}\left(e^{t\lambda_1}\left(-\frac{1}{\theta^B}\right)+e^{-t\lambda_1}\left(-\frac{1}{\theta^B}\right)\right)=\frac{1-\cosh(t\lambda_1)}{\theta^B}<0, &\text{(B.4)}\\
\frac{\partial K_t^B}{\partial K_0^B} &= \frac{1}{2}\left(e^{t\lambda_1}+e^{-t\lambda_1}\right)=\cosh(t\lambda_1)>0, &\text{(B.5)}\\
\frac{\partial K_t^B}{\partial K_0^R} &= \frac{1}{2}\left(e^{t\lambda_1}\left(v_1\right)+e^{-t\lambda_1}\left(-v_1\right)\right)=v_1\sinh(t\lambda_1)<0, &\text{(B.6)}
\end{aligned}$$

where cosh and sinh are the hyperbolic cosine and sine functions, respectively. The derivatives of K_t^R are

$$\begin{aligned}
\frac{\partial K_t^R}{\partial X^B} &= \frac{1}{\theta^R}+\frac{1}{2v_1}\left(e^{t\lambda_1}\left(-\frac{v_1}{\theta^R}\right)-e^{-t\lambda_1}\left(\frac{v_1}{\theta^R}\right)\right)=\frac{1-\cosh(t\lambda_1)}{\theta^R}<0, &\text{(B.7)}\\
\frac{\partial K_t^R}{\partial X^R} &= \frac{1}{2v_1}\left(e^{t\lambda_1}\left(-\frac{1}{\theta^B}\right)-e^{-t\lambda_1}\left(-\frac{1}{\theta^B}\right)\right)=\frac{\sinh(t\lambda_1)}{\lambda_1}>0, &\text{(B.8)}\\
\frac{\partial K_t^R}{\partial K_0^B} &= \frac{1}{2v_1}\left(e^{t\lambda_1}-e^{-t\lambda_1}\right)=\frac{\sinh(t\lambda_1)}{v_1}<0, &\text{(B.9)}\\
\frac{\partial K_t^R}{\partial K_0^R} &= \frac{1}{2v_1}\left(e^{t\lambda_1}\left(v_1\right)-e^{-t\lambda_1}\left(-v_1\right)\right)=\cosh(t\lambda_1)>0. &\text{(B.10)}
\end{aligned}$$

The following relationships are useful:

$$\begin{aligned}
\mathcal{A}\mathcal{B} &= \left(\tilde{K}_0^B\right)^2-\left(v_1\tilde{K}_0^R\right)^2, &\text{(B.11)}\\
\mathcal{A}-\mathcal{B} &= 2v_1\tilde{K}_0^R, &\text{(B.12)}\\
\mathcal{A}+\mathcal{B} &= 2\tilde{K}_0^B. &\text{(B.13)}
\end{aligned}$$

Casualties Red casualties are $D_t^R=\int_0^t\theta^BK_u^Bdu$, that is

$$\begin{aligned}
D_t^R &= \theta^B\int_0^t\left[\tilde{K}_u^B+\bar{K}^B\right]du,\\
&= \theta^B\frac{1}{2}\int_0^t\left[e^{u\lambda_1}\mathcal{A}+e^{u\lambda_2}\mathcal{B}\right]du+tX^R,\\
&= \frac{\theta^B}{2\lambda_1}\left[\mathcal{A}\left(e^{t\lambda_1}-1\right)+\mathcal{B}\left(1-e^{-t\lambda_1}\right)\right]+tX^R.
\end{aligned}$$

Using $\theta^B/\lambda_1=-1/v_1$, Equation (B.1), and Equation (B.12) yields

$$\begin{aligned}
D_t^R &= tX^R-\frac{1}{2v_1}\left[\mathcal{A}e^{t\lambda_1}-(\mathcal{A}-\mathcal{B})-\mathcal{B}e^{-t\lambda_1}\right],\\
&= tX^R+K_0^R-K_t^R. &\text{(B.14)}
\end{aligned}$$

and Blue casualties are $D_t^B = \int_0^t \theta^R K_d^R u$:

$$
\begin{aligned}
D_t^B &= \theta^R \int_0^t \left[\tilde{K}_u^R + \bar{K}^R\right] du, \\
&= \theta^R \frac{1}{2v_1}\left[\mathcal{A}\int_0^t e^{u\lambda_1} du - \mathcal{B}\int_0^t e^{u\lambda_2} du\right] + tX^B, \\
&= \theta^R \frac{1}{2v_1}\frac{1}{\lambda_1}\left[\mathcal{A}\left(e^{t\lambda_1} - 1\right) - \mathcal{B}\left(1 - e^{-t\lambda_1}\right)\right] + tX^B.
\end{aligned}
$$

Using $v_1\lambda_1 = -\theta^R$, Equation (B.2), and Equation (B.13) yields

$$
\begin{aligned}
D_t^B &= tX^B - \frac{1}{2}\left[\mathcal{A}e^{t\lambda_1} - (\mathcal{A} + \mathcal{B}) + \mathcal{B}e^{-t\lambda_1}\right], \\
&= tX^B + K_0^B - K_t^B. \qquad \text{(B.15)}
\end{aligned}
$$

C Time to military conclusion

Assume that $\mathcal{B} > 0$ such that, eventually, $K_\tau^R = 0$. Implicit differentiation yields

$$
\frac{\partial \tau}{\partial X^B} = -\frac{\partial K_\tau^R / \partial X^B}{\partial K_\tau^R / \partial \tau} = \frac{(1 - \cosh(\tau\lambda_1))/\theta^R}{\theta^B \tilde{K}_\tau^B} < 0, \qquad \text{(C.1)}
$$

$$
\frac{\partial \tau}{\partial X^R} = -\frac{\partial K_\tau^R / \partial X^R}{\partial K_\tau^R / \partial \tau} = \frac{\sinh(\tau\lambda_1)/\lambda_1}{\theta^B \tilde{K}_\tau^B} > 0, \qquad \text{(C.2)}
$$

$$
\frac{\partial \tau}{\partial K_0^B} = -\frac{\partial K_\tau^R / \partial K_0^B}{\partial K_\tau^R / \partial \tau} = \frac{\sinh(\tau\lambda_1)/v_1}{\theta^B \tilde{K}_\tau^B} < 0, \qquad \text{(C.3)}
$$

$$
\frac{\partial \tau}{\partial K_0^R} = -\frac{\partial K_\tau^R / \partial K_0^R}{\partial K_\tau^R / \partial \tau} = \frac{\cosh(\tau\lambda_1)}{\theta^B \tilde{K}_\tau^B} > 0. \qquad \text{(C.4)}
$$

A level curve of τ in the (K_0^R, K_0^B) plane is defined by

$$
0 = \frac{\partial \tau}{\partial K_0^B} dK_0^B + \frac{\partial \tau}{\partial K_0^R} dK_0^R \qquad \Rightarrow \qquad \left.\frac{dK_0^B}{dK_0^R}\right|_{d\tau=0} = -v_1 \frac{\cosh(\tau\lambda_1)}{\sinh(\tau\lambda_1)}.
$$

Similarly, a level curve of τ in the (X^R, X^B) plane is defined by

$$
0 = \frac{\partial \tau}{\partial X^B} dX^B + \frac{\partial \tau}{\partial X^R} dX^R \qquad \Rightarrow \qquad \left.\frac{dX^B}{dX^R}\right|_{d\tau=0} = v_1 \frac{\sinh(\tau\lambda_1)}{1 - \cosh(\tau\lambda_1)}.
$$

D End-of-war capital

Equations (B.1) and (B.2) imply

$$\begin{aligned} 4\left(v_1 \tilde{K}_t^R\right)^2 &= e^{2t\lambda_1}\mathcal{A}^2 + e^{2t\lambda_2}\mathcal{B}^2 - 2\mathcal{A}\mathcal{B}, \\ 4\left(\tilde{K}_t^B\right)^2 &= e^{2t\lambda_1}\mathcal{A}^2 + e^{2t\lambda_2}\mathcal{B}^2 + 2\mathcal{A}\mathcal{B}. \end{aligned}$$

Adding up yields

$$\left(\tilde{K}_t^B\right)^2 - \left(v_1 \tilde{K}_t^R\right)^2 = \mathcal{A}\mathcal{B}. \tag{D.1}$$

Note that the result above is also true at date 0 (Equation B.11): The difference on the left-hand side is constant throughout the duration of the war. It must then hold at date τ when $K_\tau^R = 0$, implying

$$K_\tau^B = \bar{K}^B + \underbrace{\sqrt{\left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}}}_{\Delta}.$$

Thus, $K_\tau^B > \bar{K}^B$ and it is immediate that $K_\tau^B > K_0^B$ whenever $\bar{K}^B > K_0^B$. Suppose now that $K_0^B > \bar{K}^B$, then $K_\tau^B > K_0^B$ whenever

$$\begin{aligned} \sqrt{\left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}} &> K_0^B - \bar{K}^B, \\ \left(v_1 \bar{K}^R\right)^2 + \left(\tilde{K}_0^B\right)^2 - \left(v_1 \tilde{K}_0^R\right)^2 &> (K_0^B - \bar{K}^B)^2, \\ 2\bar{K}^R &> K_0^R, \end{aligned}$$

where the second line follows from Equation (B.11). Then

$$\begin{aligned} \frac{\partial K_\tau^B}{\partial X^B} &= \frac{\partial}{\partial X^B}\left[\left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}\right]^{1/2}, \\ &= \frac{1}{2}\left[\left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}\right]^{-1/2} \frac{\partial}{\partial X^B}\left[\left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}\right], \\ &= \frac{1}{\theta^B}\frac{K_0^R}{K_\tau^B - \bar{K}^B}, \end{aligned} \tag{D.2}$$

$$\begin{aligned} \frac{\partial K_\tau^B}{\partial X^R} &= \frac{1}{\theta^B} + \frac{1}{2}\left(\left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}\right)^{1/2-1} \frac{\partial}{\partial X^R}\left(\left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}\right), \\ &= \frac{1}{\theta^B} - \frac{1}{2}\left(\Delta^2\right)^{-1/2} 2\left(K_0^B - \bar{K}^B\right)\frac{1}{\theta^B}, \\ &= \frac{1}{\theta^B}\frac{K_\tau^B - K_0^B}{K_\tau^B - \bar{K}^B}. \end{aligned} \tag{D.3}$$

$$
\begin{aligned}
\frac{\partial K_\tau^B}{\partial K_0^B} &= \frac{\partial}{\partial K_0^B}\left[\left(v_1\bar{K}^R\right)^2+\mathcal{AB}\right]^{1/2}, \\
&= \frac{1}{2}\left[\left(v_1\bar{K}^R\right)^2+\mathcal{AB}\right]^{-1/2}\frac{\partial}{\partial K_0^B}\left[\left(v_1\bar{K}^R\right)^2+\mathcal{AB}\right], \\
&= \frac{K_0^B-\bar{K}^B}{K_\tau^B-\bar{K}^B},
\end{aligned} \tag{D.4}
$$

and

$$
\begin{aligned}
\frac{\partial K_\tau^B}{\partial K_0^R} &= \frac{1}{2}\left(\left(v_1\bar{K}^R\right)^2+\mathcal{AB}\right)^{1/2-1}\frac{\partial}{\partial K_0^R}\left(\left(v_1\bar{K}^R\right)^2+\mathcal{AB}\right), \\
&= \frac{1}{2}\left(\Delta^2\right)^{-1/2}\left(-v_1^2 2K_0^R+2v_1^2\bar{K}^R\right), \\
&= -v_1^2\frac{K_0^R-\bar{K}^R}{K_\tau^B-\bar{K}^B},
\end{aligned} \tag{D.5}
$$

E Casualties at military conclusion

Assume that $\mathcal{B}>0$ such that, eventually, $K_\tau^R=0$. The derivatives of D_τ^R are

$$
\begin{aligned}
\frac{\partial D_\tau^R}{\partial X^B} &= \theta^B\int_0^\tau \frac{\partial K_u^B}{\partial X^B}du = \theta^B K_\tau^B\frac{\partial \tau}{\partial X^B}+\theta^B\int_0^\tau\frac{\partial K_u^B}{\partial X^B}du \\
&= \theta^B K_\tau^B\frac{1-\cosh\left(\tau\lambda_1\right)}{\theta^R\theta^B\tilde{K}_\tau^B}+\theta^B\int_0^\tau\frac{\sinh\left(u\lambda_1\right)}{\lambda_1}du, \\
&= \frac{K_\tau^B}{\tilde{K}_\tau^B}\frac{1-\cosh\left(\tau\lambda_1\right)}{\theta^R}+\frac{\cosh\left(\tau\lambda_1\right)-1}{\theta^R}=\frac{1-\cosh\left(\tau\lambda_1\right)}{\theta^R}\left(\frac{\bar{K}^B}{\tilde{K}_\tau^B}\right)<0,
\end{aligned} \tag{E.1}
$$

$$
\begin{aligned}
\frac{\partial D_\tau^R}{\partial X^R} &= \theta^B\int_0^\tau\frac{\partial K_u^B du}{\partial X^R}=\theta^B K_\tau^B\frac{\partial\tau}{\partial X^R}+\theta^B\int_0^\tau\frac{\partial K_u^B}{\partial X^R}du, \\
&= \theta^B K_\tau^B\frac{\sinh\left(\tau\lambda_1\right)}{\theta^B\tilde{K}_\tau^B\lambda_1}+\theta^B\int_0^\tau\frac{1-\cosh\left(u\lambda_1\right)}{\theta^B}du, \\
&= \frac{K_\tau^B}{\tilde{K}_\tau^B}\frac{\sinh\left(\tau\lambda_1\right)}{\lambda_1}+\tau-\frac{\sinh\left(\tau\lambda_1\right)}{\lambda_1}=\tau+\frac{\sinh\left(\tau\lambda_1\right)}{\lambda_1}\frac{\bar{K}^B}{\tilde{K}_\tau^B}>0,
\end{aligned} \tag{E.2}
$$

$$
\begin{aligned}
\frac{\partial D_\tau^R}{\partial K_0^B} &= \theta^B\int_0^\tau\frac{\partial K_u^B du}{\partial K_0^B}=\theta^B K_\tau^B\frac{\partial\tau}{\partial K_0^B}+\theta^B\int_0^\tau\frac{\partial K_u^B}{\partial K_0^B}du, \\
&= \theta^B K_\tau^B\frac{\sinh\left(\tau\lambda_1\right)}{\theta^B\tilde{K}_\tau^B v_1}+\theta^B\int_0^\tau\cosh\left(u\lambda_1\right)du, \\
&= \frac{K_\tau^B}{\tilde{K}_\tau^B}\frac{\sinh\left(\tau\lambda_1\right)}{v_1}+\frac{\theta^B}{\lambda_1}\sinh\left(\tau\lambda_1\right)=\frac{\sinh\left(\tau\lambda_1\right)}{v_1}\frac{\bar{K}^B}{\tilde{K}_\tau^B}<0,
\end{aligned} \tag{E.3}
$$

and

$$\begin{aligned}
\frac{\partial D_\tau^R}{\partial K_0^R} &= \theta^B \int_0^\tau \frac{\partial K_u^B du}{\partial K_0^R} = \theta^B K_\tau^B \frac{\partial \tau}{\partial K_0^R} + \theta^B \int_0^\tau \frac{\partial K_u^B}{\partial K_0^R} du, \\
&= \theta^B K_\tau^B \frac{\cosh(\tau\lambda_1)}{\theta^B \tilde{K}_\tau^B} + \theta^B v_1 \int_0^\tau \sinh(u\lambda_1)\, du, \\
&= \frac{K_\tau^B}{\tilde{K}_\tau^B} \cosh(\tau\lambda_1) + \frac{\theta^B v_1}{\lambda_1} (\cosh(\tau\lambda_1) - 1) = 1 + \cosh(\tau\lambda_1) \frac{\bar{K}^B}{\tilde{K}_\tau^B} > 0. \qquad \text{(E.4)}
\end{aligned}$$

The derivatives of D_τ^B are

$$\begin{aligned}
\frac{\partial D_\tau^B}{\partial X^B} &= \theta^R \int_0^\tau \frac{\partial K_u^R}{\partial X^B} du = \theta^R \int_0^\tau \frac{1 - \cosh(u\lambda_1)}{\theta^R} du = \tau - \int_0^\tau \cosh(u\lambda_1)\, du, \\
&= \tau - \frac{\sinh(\tau\lambda_1)}{\lambda_1} < 0, \qquad \text{(E.5)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial D_\tau^B}{\partial X^R} &= \theta^R \int_0^\tau \frac{\partial K_u^R}{\partial X^R} du = \theta^R \int_0^\tau \frac{\sinh(u\lambda_1)}{\lambda_1} du = \frac{\theta^R}{\lambda_1^2} (\cosh(\tau\lambda_1) - 1), \\
&= \frac{\cosh(\tau\lambda_1) - 1}{\theta^B} > 0, \qquad \text{(E.6)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial D_\tau^B}{\partial K_0^B} &= \theta^R \int_0^\tau \frac{\partial K_u^R}{\partial K_0^B} du = \theta^R \int_0^\tau \frac{\sinh(u\lambda_1)}{v_1} du = \frac{\theta^R}{v_1\lambda_1} (\cosh(\tau\lambda_1) - 1), \\
&= 1 - \cosh(\tau\lambda_1) < 0, \qquad \text{(E.7)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial D_\tau^B}{\partial K_0^R} &= \theta^R \int_0^\tau \frac{\partial K_u^R}{\partial K_0^R} du = \theta^R \int_0^\tau \cosh(u\lambda_1)\, du = \theta^R \sinh(\tau\lambda_1)/\lambda_1, \\
&= -v_1 \sinh(\tau\lambda_1) > 0. \qquad \text{(E.8)}
\end{aligned}$$

F Political conclusion

Blue reaches the threshold $\bar{D}^B$ at date τ^B such that (using B.15)

$$\tau^B X^B + K_0^B - K_{\tau^B}^B = \bar{D}^B.$$

Implicitly differentiating with respect to τ^B and $\bar{D}^B$ yields

$$\frac{\partial \tau^B}{\partial \bar{D}^B} = \frac{1}{\theta^R K_{\tau^B}^R} > 0,$$

and, further differentiating,

$$\frac{\partial \tau^B}{\partial X^B} = \frac{\partial \tau^B}{\partial \bar{D}^B} \times \left(\frac{\sinh(\tau^B \lambda_1)}{\lambda_1} - \tau^B\right) > 0, \tag{F.1}$$

$$\frac{\partial \tau^B}{\partial X^R} = \frac{\partial \tau^B}{\partial \bar{D}^B} \times \frac{1 - \cosh(\tau^B \lambda_1)}{\theta^B} < 0, \tag{F.2}$$

$$\frac{\partial \tau^B}{\partial K_0^B} = \frac{\partial \tau^B}{\partial \bar{D}^B} \times \left(\cosh(\tau^B \lambda_1) - 1\right) > 0, \tag{F.3}$$

$$\frac{\partial \tau^B}{\partial K_0^R} = \frac{\partial \tau^B}{\partial \bar{D}^B} \times v_1 \sinh(\tau^B \lambda_1) < 0. \tag{F.4}$$

The level curves of τ^B in the (K_0^R, K_0^B) plane are defined by

$$0 = \frac{\partial \tau^B}{\partial K_0^R} dK_0^R + \frac{\partial \tau^B}{\partial K_0^B} dK_0^B \qquad \Rightarrow \qquad \left.\frac{dK_0^B}{dK_0^R}\right|_{d\tau^B=0} = -v_1 \frac{\sinh\left(\tau^B \lambda_1\right)}{\cosh\left(\tau^B \lambda_1\right) - 1},$$

and the level curves in the (X^R, X^B) plane are

$$0 = \frac{\partial \tau^B}{\partial X^R} dX^R + \frac{\partial \tau^B}{\partial X^B} dX^B \qquad \Rightarrow \qquad \left.\frac{dX^B}{dX^R}\right|_{d\tau^B=0} = v_1 \frac{1 - \cos\left(\tau^B \lambda_1\right)}{\sinh\left(\tau^B \lambda_1\right) - \tau^B \lambda_1}.$$

Similarly, Red crosses its threshold $\bar{D}^R$ at date τ^R such that (using B.14)

$$\tau^R X^R + K_0^R - K_{\tau^R}^R = \bar{D}^R.$$

This implies

$$\frac{\partial \tau^R}{\partial \bar{D}^R} = \frac{1}{\theta^B K_t^B} > 0,$$

and

$$\frac{\partial \tau^R}{\partial X^B} = \frac{\partial \tau^R}{\partial \bar{D}^R} \times \frac{1 - \cosh\left(\tau^R \lambda_1\right)}{\theta^R} < 0, \tag{F.5}$$

$$\frac{\partial \tau^R}{\partial X^R} = \frac{\partial \tau^R}{\partial \bar{D}^R} \times \left(\frac{\sinh\left(\tau^R \lambda_1\right)}{\lambda_1} - \tau^R\right) > 0, \tag{F.6}$$

$$\frac{\partial \tau^R}{\partial K_0^B} = \frac{\partial \tau^R}{\partial \bar{D}^R} \times \frac{\sinh\left(\tau^R \lambda_1\right)}{v_1} < 0, \tag{F.7}$$

$$\frac{\partial \tau^R}{\partial K_0^R} = \frac{\partial \tau^R}{\partial \bar{D}^R} \times \left(\cosh\left(\tau^R \lambda_1\right) - 1\right) > 0. \tag{F.8}$$

G Casualties at political conclusion

Red casualties when Red initiates a political conclusion are given by $D_{\tau^R}^R = \bar{D}^R$ and Blue casualties when Blue initiates a political conclusion are given by $D_{\tau^B}^B = \bar{D}^B$.

Blue casualties when Red initiates a political conclusion

$$\begin{aligned}
\frac{\partial D^B_{\tau^R}}{\partial X^B} &= \frac{\partial}{\partial X^B}\left(\theta^R \int_0^{\tau^R} K^R_u du\right) = \theta^R K^R_{\tau^R}\frac{\partial \tau^R}{\partial X^B} + \theta^R \int_0^{\tau^R} \frac{\partial}{\partial X^B} K^R_u du \\
&= \theta^R K^R_{\tau^R}\frac{1}{\theta^B K^B_{\tau^R}}\frac{1-\cosh\left(\tau^R\lambda_1\right)}{\theta^R} + \theta^R \int_0^{\tau^R} \frac{1-\cosh\left(u\lambda_1\right)}{\theta^R} du \\
&= \underbrace{\frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}}\frac{1-\cosh\left(\tau^R\lambda_1\right)}{\theta^R}}_{<0} + \underbrace{\tau^R - \frac{\sinh\left(\tau^R\lambda_1\right)}{\lambda_1}}_{<0} < 0,
\end{aligned}$$

where the second line uses Equations (F.5) and (B.7).

$$\begin{aligned}
\frac{\partial D^B_{\tau^R}}{\partial X^R} &= \frac{\partial}{\partial X^R}\left(\theta^R \int_0^{\tau^R} K^R_u du\right) = \theta^R K^R_{\tau^R}\frac{\partial \tau^R}{\partial X^R} + \theta^R \int_0^{\tau^R} \frac{\partial}{\partial X^R} K^R_u du \\
&= \frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}}\left(\frac{\sinh\left(\tau^R\lambda_1\right)}{\lambda_1} - \tau^R\right) + \theta^R \int_0^{\tau^R} \frac{\sinh\left(u\lambda_1\right)}{\lambda_1} du \\
&= \underbrace{\frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}}\left(\frac{\sinh\left(\tau^R\lambda_1\right)}{\lambda_1} - \tau^R\right)}_{>0} + \underbrace{\frac{1}{\theta^B}\left(\cosh\left(\tau^R\lambda_1\right) - 1\right)}_{>0} > 0,
\end{aligned}$$

where the second line uses Equations (F.6) and (B.8).

$$\begin{aligned}
\frac{\partial D^B_{\tau^R}}{\partial K^B_0} &= \frac{\partial}{\partial K^B_0}\left(\theta^R \int_0^{\tau^R} K^R_u du\right) = \theta^R K^R_{\tau^R}\frac{\partial \tau^R}{\partial K^B_0} + \theta^R \int_0^{\tau^R} \frac{\partial}{\partial K^B_0} K^R_u du \\
&= \theta^R K^R_{\tau^R}\frac{1}{\theta^B K^B_{\tau^R}}\frac{\sinh\left(\tau^R\lambda_1\right)}{v_1} + \theta^R \int_0^{\tau^R} \frac{\sinh\left(u\lambda_1\right)}{v_1} du \\
&= \underbrace{\frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}}\frac{\sinh\left(\tau^R\lambda_1\right)}{v_1}}_{<0} + \underbrace{1 - \cosh\left(\tau^R\lambda_1\right)}_{<0} < 0,
\end{aligned}$$

where the second line uses Equations (F.7) and (B.9).

$$\begin{aligned}
\frac{\partial D^B_{\tau^R}}{\partial K^R_0} &= \frac{\partial}{\partial K^R_0}\left(\theta^R \int_0^{\tau^R} K^R_u du\right) = \theta^R K^R_{\tau^R}\frac{\partial \tau^R}{\partial K^R_0} + \theta^R \int_0^{\tau^R} \frac{\partial}{\partial K^R_0} K^R_u du \\
&= \frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}}\left(\cosh\left(\tau^R\lambda_1\right) - 1\right) + \theta^R \int_0^{\tau^R} \cosh\left(u\lambda_1\right) du \\
&= \underbrace{\frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}}\left(\cosh\left(\tau^R\lambda_1\right) - 1\right)}_{>0} + \underbrace{\frac{\theta^R}{\lambda_1}\sinh\left(\tau^R\lambda_1\right)}_{>0} > 0,
\end{aligned}$$

where the second line uses Equations (F.8) and (B.10).

Red casualties when Blue initiates a political conclusion

$$
\begin{aligned}
\frac{\partial D^R_{\tau^B}}{\partial X^B} &= \frac{\partial}{\partial X^B}\left(\theta^B \int_0^{\tau^B} K^B_u du\right) = \theta^B K^B_{\tau^B}\frac{\partial \tau^B}{\partial X^B} + \theta^B \int_0^{\tau^B} \frac{\partial}{\partial X^B} K^B_u du, \\
&= \theta^B K^B_{\tau^B}\frac{1}{\theta^R K^R_{\tau^B}}\left(\frac{\sinh\left(\tau^B\lambda_1\right)}{\lambda_1} - \tau^B\right) + \theta^B \int_0^{\tau^B} \frac{\sinh\left(u\lambda_1\right)}{\lambda_1} du, \\
&= \underbrace{\frac{\theta^B K^B_{\tau^B}}{\theta^R K^R_{\tau^B}}\left(\frac{\sinh\left(\tau^B\lambda_1\right)}{\lambda_1} - \tau^B\right)}_{>0} + \underbrace{\frac{\cosh\left(\tau^B\lambda_1\right)-1}{\theta^R}}_{>0} > 0,
\end{aligned}
$$

where the second line uses Equations (F.1) and (B.3).

$$
\begin{aligned}
\frac{\partial D^R_{\tau^B}}{\partial X^R} &= \frac{\partial}{\partial X^R}\left(\theta^B \int_0^{\tau^B} K^B_u du\right) = \theta^B K^B_{\tau^B}\frac{\partial \tau^B}{\partial X^R} + \theta^B \int_0^{\tau^B} \frac{\partial}{\partial X^R} K^B_u du, \\
&= \theta^B K^B_{\tau^B}\frac{1}{\theta^R K^R_{\tau^B}}\left(\frac{1-\cosh\left(\tau^B\lambda_1\right)}{\theta^B}\right) + \theta^B \int_0^{\tau^B} \frac{1-\cosh\left(u\lambda_1\right)}{\theta^B} du, \\
&= \underbrace{\frac{\theta^B K^B_{\tau^B}}{\theta^R K^R_{\tau^B}}\left(\frac{1-\cosh\left(\tau^B\lambda_1\right)}{\theta^B}\right)}_{<0} + \underbrace{\tau^B - \frac{\sinh\left(\tau^B\lambda_1\right)}{\lambda_1}}_{<0} < 0,
\end{aligned}
$$

where the second line uses Equations (F.2) and (B.4).

$$
\begin{aligned}
\frac{\partial D^R_{\tau^B}}{\partial K^B_0} &= \frac{\partial}{\partial K^B_0}\left(\theta^B \int_0^{\tau^B} K^B_u du\right) = \theta^B K^B_{\tau^B}\frac{\partial \tau^B}{\partial K^B_0} + \theta^B \int_0^{\tau^B} \frac{\partial}{\partial K^B_0} K^B_u du, \\
&= \theta^B K^B_{\tau^B}\frac{1}{\theta^R K^R_{\tau^B}}\left(\cosh\left(\tau^B\lambda_1\right)-1\right) + \theta^B \int_0^{\tau^B} \cosh\left(u\lambda_1\right) du, \\
&= \underbrace{\frac{\theta^B K^B_{\tau^B}}{\theta^R K^R_{\tau^B}}\left(\cosh\left(\tau^B\lambda_1\right)-1\right)}_{>0} - \underbrace{\frac{\sinh\left(\tau^B\lambda_1\right)}{v_1}}_{<0} > 0,
\end{aligned}
$$

where the second line uses Equations (F.3) and (B.5).

$$
\begin{aligned}
\frac{\partial D^R_{\tau^B}}{\partial K^R_0} &= \frac{\partial}{\partial K^R_0}\left(\theta^B \int_0^{\tau^B} K^B_u du\right) = \theta^B K^B_{\tau^B}\frac{\partial \tau^B}{\partial K^R_0} + \theta^B \int_0^{\tau^B} \frac{\partial}{\partial K^R_0} K^B_u du, \\
&= \theta^B K^B_{\tau^B}\frac{1}{\theta^R K^R_{\tau^B}} v_1 \sinh\left(\tau^B\lambda_1\right) + \theta^B \int_0^{\tau^B} v_1 \sinh\left(u\lambda_1\right) du, \\
&= \frac{\theta^B K^B_{\tau^B}}{\theta^R K^R_{\tau^B}} v_1 \sinh\left(\tau^B\lambda_1\right) + \frac{\theta^B v_1}{\lambda_1}\left(\cosh\left(\tau^B\lambda_1\right)-1\right), \\
&= \underbrace{\frac{\theta^B K^B_{\tau^B}}{\theta^R K^R_{\tau^B}} v_1 \sinh\left(\tau^B\lambda_1\right)}_{<0} + \underbrace{1-\cosh\left(\tau^B\lambda_1\right)}_{<0} < 0,
\end{aligned}
$$

where the second line uses Equations (F.4) and (B.6).

H Civilians

Suppose Blue allocates a fraction $\alpha^B \in (0,1)$ of its weapons stock to the destruction of Red weapons, and the remainder to the destruction of Red civilian resources. I assume civilian resources to be combinations of human and material resources, like weapons are. I further assume a constant rate of transformation, η^R, from civilian resources to weapons. Let $\alpha^R \in (0,1)$ and η^B have symmetric interpretations. Equations (1) and (2) become

$$\begin{aligned} dK_t^R/dt &= -\theta^B \alpha^B K_t^B + X^R, \\ dK_t^B/dt &= -\theta^R \alpha^R K_t^R + X^B. \end{aligned}$$

The flow of Red civilian resources destroyed by Blue weapons at t, expressed in Red weapons, is then $\eta^R \theta^B (1-\alpha^B) K_t^B$. Red casualties becomes

$$D_t^R = \alpha^B \theta^B \int_0^t K_u^B du + \eta^R \theta^B (1-\alpha^B) \int_0^t K_u^B du,$$

where the first element on the right-hand side is the casualties from fighting and the second element is the casualties from attacks on civilian resources. It follows that

$$\begin{aligned} D_t^R &= \left(1 + \eta^R \frac{1-\alpha^B}{\alpha^B}\right) \alpha^B \theta^B \int_0^t K_u^B du, \\ &= \left(1 + \eta^R \frac{1-\alpha^B}{\alpha^B}\right) \left(tX^R + K_0^R - K_t^R\right), \end{aligned}$$

where the last line follows from the same derivation as with Equation (B.14). Similarly, Blue casualties are

$$D_t^B = \left(1 + \eta^B \frac{1-\alpha^R}{\alpha^R}\right) \left(tX^B + K_0^B - K_t^B\right).$$

There are two differences between this model and the one I analyze in the main body of the paper. First, the laws of motion of the weapons stock are as in Equations (1) and (2) with modified attrition coefficients, that is $\theta^R \alpha^R$ instead of θ^R and $\theta^B \alpha^B$ instead of θ^B. Second, casualties are scaled versions of that in Equations (8) and (9). Thus, this model is isomorphic to the model in the main body of the paper and the analysis there remains valid with this representation of civilian casualties.